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A New Multidimensional Half-Discrete Reverse Hardy-Hilbert's Inequality with One Partial Sum

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Abstract

By means of the weight functions, the idea of introduced parameters and the techniques of real analysis, a multidimensional halfdiscrete reverse Hardy-Hilbert's inequality with one partial sum is obtained. The equivalent statements of the best value related to parameters are considered, and some corollaries are deduced.

Keywords: weight function; parameter; Beta function; partial sum; multidimensional half-discrete Hardy-Hilbert's inequality; reverse

Introduction

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, we have the following Hardy-Hilbert′s inequality with the best value $\frac{\pi}{\sin(\pi/p)}$ (cf [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

Setting $f(x), g(y) \geq 0,\ 0 < \int_0^{\infty} f^p(x)dx < \infty$ and $0 < \int_0^{\infty} g^q(y)dy < \infty$, we have the integral analogue of (1) with the same best value named in Hardy-Hilbert's integral inequality as follows (cf [1], Theorem 316):

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}. \tag{2}$$

In 2006, by means of Euler-Maclaorin summation formula, Krnić et al. [2] gave an extension of (1) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right)^{\frac{1}{q}}. \tag{3}$$

where, $\lambda_1, \lambda_2 \in (0, 2], \lambda_1 + \lambda_2 = \lambda \in (0, 4]$, the constant $B(\lambda_1, \lambda_2)$ is the best value, and

$$B(u, v) = \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt, u, v > 0 \tag{4}$$

is the Beta function. In 2019, by means of (3) and Abel's partial summation formula, Adiyasuren et al. [3] obtained an extended application of (3) involving two partial sums. In 2020, Mo et al. [4] gave an extension of (2) involving two upper limit functions. Inequalities (1)-(2) with their extensions played an important role in analysis and its applications (cf. [5]-[15]).

In 2016-2017, Hong et al. [16]-[17] considered several equivalent conditions of the extensions of (1) and (2) with a few parameters related to the best values. Some other results were provided by [18]-[20]. In 2023, Hong et al. [21] gave a more accurate multidimensional half-discrete Hilbert-type inequality involving one derivative function of m-order, and [22] gave an extended inequality with the same kernel involving one multiple upper limit function. Some dependent results were published by [23]-[27].

In this paper, following the way of [21] and [22], by using the weight functions, the idea of introduced μarameters and the techniques of real analysis. a multidimensional half-discrete reverse Hardy-Hilbert's inequality with the new kernel as $\frac{1}{(u(m)+||y||_{\beta}^{\alpha})^{\lambda+i}}$ $(\alpha, \lambda > 0, i \in \{0, 1\})$ with one partial sum is obtained. The equivalent statements of the best value related to parameters are considered, and some corollaries are provided.

Some Lemmas

In what follows, we assume that
(H1). $p<0$ $(0<q<1), \frac{1}{p}+\frac{1}{q}=1, \alpha, \beta \in \mathbf{R}_{+}:=(0,\infty)$, $\lambda>0$, $\lambda_1, \lambda_2 \in (0,\lambda), m_0, n \in \mathbf{N}:=\{1,2,\cdots\}, u(x), u'(x)>0, u''(x)\leq 0$, there exists a constant $\eta_0<1$, such that $(u(x))^{\lambda_1-\eta_0}u'(x)$ is decreasing in $x \in (m_0-1,\infty)$, with $u(\infty)=\infty$, $\widehat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widehat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, $a_k \geq 0$, $A_m^{(0)}:=a_m$, $A_m^{(1)}:=\sum_{k=m_0}^{m} a_k$ $(k,m \in \mathbf{N}_{m_0}:=\{m_0,m_0+1,\cdots\})$, satisfying $A_m^{(1)}=o(e^{tu(m)})$ $(t>0; m\to\infty)$, and

$$0<\sum_{m=m_0}^{\infty}\frac{(u'(m))^{1-p}}{(u(m))^{p(\widehat{\lambda}_1-1)+1}}a_m^p<\infty.$$

For $g(y)\geq 0$, $y=(y_1,\cdots,y_n)\in \mathbf{R}_+^n$, $||y||_\beta:=(\sum_{i=1}^n y_i^\beta)^{\frac{1}{\beta}}$, we still have

$$0<\int_{\mathbf{R}_+^n}||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n}g^q(y)dy<\infty.$$

Remark 1. (i) For $\gamma\in(0,1], m_0=1, u(x)=x^\gamma, x\in(0,\infty)$, $\lambda_1 \in (0,\frac{1}{\gamma}), u(x)>0, u'(x)=\gamma x^{\gamma-1}>0, u''(x)=\gamma(\gamma-1)x^{\gamma-2}<0, u(\infty)=\lim_{x\to\infty}x^\gamma=\infty$, $\eta_0\in[\lambda_1-\frac{1}{\gamma}+1,1), (u(x))^{\lambda_1-\eta_0}u'(x)=\gamma x^{(\lambda_1-\eta_0+1)\gamma-1}$ is decreasing in $x\in(0,\infty)$.

(ii) For $\gamma\in(0,1], m_0=2, u(x)=\ln^\gamma x, x\in(1,\infty)$, $\lambda_1\in(0,\frac{1}{\gamma})$, $u(x)>0$, $u'(x)=\frac{\gamma}{x}\ln^{\gamma-1}x>0, u''(x)<0$, $u(\infty)=\lim_{x\to\infty}\ln^\gamma x=\infty$, $\eta_0\in[\lambda_1-\frac{1}{\gamma}+1,1)$, $(u(x))^{\lambda_1-\eta_0}u'(x)=\frac{\gamma}{x}\ln^{(\lambda_1-\eta_0+1)\gamma-1}x$ is decreasing in $x\in(1,\infty)$.

If $M>0, \psi(u)$ $(u>0)$ is a nonnegative measurable function, then we have the following transfer formula (cf. [5], (9.1.5)):

$$\begin{aligned}&\int\cdots\int_{\{y\in\mathbf{R}_+^n;0<\sum_{i=1}^n(\frac{y_i}{M})^\beta\leq 1\}}\psi(\sum_{i=1}^n(\frac{y_i}{M})^\beta)dy_1\cdots dy_n\\ &= \frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})}\int_0^1\psi(u)u^{\frac{n}{\beta}-1}du. \qquad (5)\end{aligned}$$

(i) For $||y||_\beta=M[\sum_{i=1}^n(\frac{y_i}{M})^\beta]^{\frac{1}{\beta}}, \psi(u)=\varphi(Mu^{\frac{1}{\beta}})$, by (5), setting $v=Mu^{\frac{1}{\beta}}$, we have

$$\begin{aligned}&\int_{\mathbf{R}_+^n}\varphi(||y||_\beta)dy\\ &= \lim_{M\to\infty}\int\cdots\int_{\{y\in\mathbf{R}_+^n;0<\sum_{i=1}^n(\frac{y_i}{M})^\beta\leq 1\}}\varphi(M[\sum_{i=1}^n(\frac{y_i}{M})^\beta]^{\frac{1}{\beta}})dy_1\cdots dy_n\\ &= \lim_{M\to\infty}\frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})}\int_0^1\varphi(Mu^{\frac{1}{\beta}})u^{\frac{n}{\beta}-1}du\\ &= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\int_0^\infty\varphi(v)v^{n-1}dv. \qquad (6)\end{aligned}$$

(ii) If $\varphi(||y||_\beta) = \varphi(Mu^{\frac{1}{\beta}}) = 0$, for $u = \sum_{i=1}^n (\frac{y_i}{M})^\beta < (\frac{b}{M})^\beta$ $(b > 0)$, i.e. $||y||_\beta = Mu^{\frac{1}{\beta}} < b$, then by (6), it follows that

$$\begin{aligned}\int_{\{y\in\mathbf{R}_+^n;||y||_\beta\geq b\}} \varphi(||y||_\beta)dy &= \lim_{M\to\infty} \frac{M^n\Gamma^n(\frac{1}{\beta})}{\beta^n\Gamma(\frac{n}{\beta})} \int_{(\frac{b}{M})^\beta}^1 \varphi(Mu^{\frac{1}{\beta}})u^{\frac{n}{\beta}-1}du \\ &= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})} \int_b^\infty \varphi(v)v^{n-1}dv. \qquad (7)\end{aligned}$$

Remark 2. For $b = 1, c \in \mathbf{R}_+, \varphi(v) = v^{-\alpha c-n}$ in (7), we have

$$\int_{\{y\in\mathbf{R}_{+;}^n||y||_\beta\geq 1\}} ||y||_\beta^{-\alpha c-n}dy = \int_1^\infty v^{-\alpha c-n}v^{n-1}dv = \frac{\Gamma^n(\frac{1}{\beta})}{\alpha c\beta^{n-1}\Gamma(\frac{n}{\beta})}. \qquad (8)$$

Lemma 1 Suppose that $s \in (0,\infty), s_1, s_2 \in (0, s)$, there exists a constant $\eta_0 < 1$, such that $(u(x))^{s_1-\eta_0}u'(x)$ is decreasing in $(m_0 - 1,\infty)$. We define the following weight functions:

$$\omega_s(s_1, y) \ : \ = ||y||_\beta^{\alpha(s-s_1)} \sum_{m=m_0}^\infty \frac{(u(m))^{s_1-1}u'(m)}{(u(m) + ||y||_\beta^\alpha)^s} \ (y \in \mathbf{R}_+^n), \qquad (9)$$

$$\varpi_s(s_2, m) \ : \ = (u(m))^{s-s_2} \int_{\mathbf{R}_+^n} \frac{||y||_\beta^{\alpha s_2-n}dy}{(u(m) + ||y||_\beta^\alpha)^s} \ (m \in \mathbf{N}_{m_0}). \qquad (10)$$

The following inequality and expression are value:

$$\omega_s(s_1, y) < B(s_1, s - s_1) \ \ (y \in \mathbf{R}_+^n), \qquad (11)$$

$$\varpi_s(s_2, m) = \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}B(s - s_2, s_2) \ \ (m \in \mathbf{N}_{m_0}). \qquad (12)$$

Proof. Since $u'(x) > 0, \eta_0 < 1$, we observe that

$$(u(x))^{s_1-1}u'(x) \ (= (u(x))^{\eta_0-1}[(u(x))^{s_1-\eta_0}u'(x)])$$

is still decreasing in $(m_0 - 1,\infty)$. In view of the decreasingness property of series, setting $v = \frac{u(x)}{||y||_\beta^\alpha}$, we find

$$\begin{aligned}\omega_s(s_1, y) &< ||y||_\beta^{\alpha(s-s_1)} \int_{m_0-1}^\infty \frac{(u(x))^{s_1-1}u'(x)}{(u(x) + ||y||_\beta^\alpha)^s}dx \\ &\leq \int_0^\infty \frac{v^{s_1-1}}{(v + 1)^s}dv = B(s_1, s - s_1),\end{aligned}$$

and then we have (11).

In (6), for $\varphi(v) = \frac{v^{\alpha s_2-n}}{(u(m)+v^\alpha)^s}$, setting $t = \frac{v^\alpha}{u(m)}$, we have

$$\begin{aligned}\varpi_s(s_2, m) &= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}(u(m))^{s-s_2} \int_0^\infty \frac{v^{\alpha s_2-n}v^{n-1}}{(u(m) + v^\alpha)^s}dv \\ &= \frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}(u(m))^{s-s_2} \int_0^\infty \frac{v^{\alpha s_2-1}}{(u(m) + v^\alpha)^s}dv \\ &= \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \int_0^\infty \frac{t^{s_2-1}}{(1 + t)^s}dt\end{aligned}$$

$$= \frac{\Gamma^n(\frac{1}{\beta})B(s_2, s-s_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} = \frac{\Gamma^n(\frac{1}{\beta})B(s-s_2, s_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})},$$

and then we have (12).

This proves the lemma. □

Lemma 2. With regards to the assumption H1, for $t > 0$, we have the following inequality:

$$\sum_{m=m_0}^{\infty} e^{-tu(m)}(u'(m))^i A_m^{(i)} \geq t^{-i} \sum_{m=m_0}^{\infty} e^{-tu(m)} a_m \ (i \in \{0,1\}). \tag{13}$$

Proof. For $i = 0$, since $a_m = A_m^{(0)}$, (13) keeps the form of an equality; for $i = 1$, since $A_m^{(1)} e^{-tu(m)} = o(1)$ $(t > 0; m \to \infty)$, by Abel's partial summation formula, we find

$$\begin{aligned}\sum_{m=m_0}^{\infty} e^{-tu(m)} a_m &= \lim_{m\to\infty} A_m^{(1)} e^{-tu(m)} + \sum_{m=m_0}^{\infty} A_m^{(1)}(e^{-tu(m)} - e^{-tu(m+1)}) \\ &= \sum_{m=m_0}^{\infty} A_m^{(1)}(e^{-tu(m)} - e^{-tu(m+1)}).\end{aligned} \tag{14}$$

We set function $f(x) := e^{-tu(x)}, x \in (m_0 - 1, \infty)$. Then we find

$$f'(x) := -te^{-tu(x)}u'(x) = -th(x),$$

where, $h(x) = e^{-tu(x)}u'(x)$ is decreasing in (m_0-1, ∞), in view of $u(x), u'(x) > 0$ and $u''(x) \leq 0$. By (14) and the differentiation mid-value theorem, there exists a $\theta_m \in (0,1)$, such that

$$\begin{aligned}\sum_{m=m_0}^{\infty} e^{-tu(m)} a_m &= -\sum_{m=m_0}^{\infty} A_m^{(1)}[f(m+1) - f(m)] \\ &= -\sum_{m=m_0}^{\infty} A_m^{(1)} f'(m+\theta_m) = t\sum_{m=m_0}^{\infty} A_m^{(1)} h(m+\theta_m) \\ &\leq t\sum_{m=m_0}^{\infty} A_m^{(1)} h(m) = t\sum_{m=m_0}^{\infty} e^{-tu(m)}u'(m)A_m^{(1)},\end{aligned}$$

and then we have (13).

This proves the lemma. □

Lemma 3. With regards to the assumption H1, we have the following inequality:

$$\begin{aligned}I_\lambda \ &:= \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{a_m g(y)}{(u(m) + ||y||_\beta^\alpha)^\lambda} dy \\ &> \left(\frac{\Gamma^n(\frac{1}{\beta})B(\lambda-\lambda_2, \lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1)\end{aligned}$$

$$\times \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p} a_m^p}{(u(m))^{p(\widehat{\lambda}_1-1)+1}}\right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n} g^q(y) dy\right]^{\frac{1}{q}}. \quad (15)$$

der's inequality (cf. [28]), we have

$$\begin{aligned}
I_\lambda &= \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{1}{(u(m)+||y||_\beta^\alpha)^\lambda} \left[\frac{(u(m))^{(1-\lambda_1)/q} a_m}{||y||_\beta^{(n-\alpha\lambda_2)/p} (u'(m))^{1/q}}\right] \\
&\quad \times \left[\frac{||y||_\beta^{(n-\alpha\lambda_2)/p} g(y)}{(u(m))^{(1-\lambda_1)/q} (u'(m))^{-1/q}}\right] dy \\
&\geq \left\{\sum_{m=m_0}^{\infty} \int_{\mathbf{R}_+^n} \frac{1}{(u(m)+||y||_\beta^\alpha)^\lambda} \frac{(u(m)^{(1-\lambda_1)(p-1)} a_m^p}{||y||_\beta^{n-\alpha\lambda_2} (u'(m))^{p-1}} dy\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{1}{(u(m)+||y||_\beta^\alpha)^\lambda} \frac{||y||_\beta^{(n-\alpha\lambda_2)(q-1)} g^q(y)}{(u(m))^{1-\lambda_1} (u'(m))^{-1}} dy\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{m=m_0}^{\infty} \left[(u(m))^{\lambda-\lambda_2} \int_{\mathbf{R}_+^n} \frac{||y||_\beta^{\alpha\lambda_2-n} dy}{(u(m)+||y||_\beta^\alpha)^\lambda}\right] \frac{(u'(m))^{1-p} a_m^p}{(u(m))^{p(\widehat{\lambda}_1-1)+1}}\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\int_{\mathbf{R}_+^n} \left[||y||_\beta^{\alpha(\lambda-\lambda_1)} \sum_{m=m_0}^{\infty} \frac{(u(m)^{-1} u'(m)}{(u(m)+||y||_\beta^\alpha)^\lambda}\right] ||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n} g^q(y) dy\right\}^{\frac{1}{q}} \\
&= \left[\sum_{m=m_0}^{\infty} \varpi_\lambda(\lambda_2, m) \frac{(u'(m))^{1-p} a_m^p}{(u(m))^{p(\widehat{\lambda}_1-1)+1}}\right]^{\frac{1}{p}} \\
&\quad \times \left[\int_{\mathbf{R}_+^n} \omega_\lambda(\lambda_1, y) ||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n} g^q(y) dy\right]^{\frac{1}{q}}.
\end{aligned}$$

By (11) and (12), for $p<0$ $(0<q<1)$, $s=\lambda>0$, $s_1=\lambda_1\in(0,\lambda)$, $s_2=\lambda_2\in(0,\lambda)$, $(u(x))^{s_1-\eta_0}u'(x)=(u(x))^{\lambda_1-\eta_0}u'(x)$ $(\eta_0<1)$ is decreasing in (m_0-1,∞), in view of H1, we have (15).

This proves the lemma. □

Main Result

Theorem 1. With regards to the assumption H1, for $i\in\{0,1\}$, we have the following multidimensional half-discrete reverse Hardy-Hilbert's inequality with one partial sum:

$$\begin{aligned}
I &:= \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{(u'(m))^i A_m^{(i)} g(y)}{(u(m)+||y||_\beta^\alpha)^{\lambda+i}} dye \\
&> \frac{1}{\lambda^i} \left(\frac{\Gamma^n(\frac{1}{\beta}) B(\lambda-\lambda_2, \lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda-\lambda_1) \\
&\quad \times \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p} a_m^p}{(u(m))^{p(\widehat{\lambda}_1-1)+1}}\right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n} g^q(y) dy\right]^{\frac{1}{q}}. \quad (16)
\end{aligned}$$

In particular, for $\lambda = \lambda_1 + \lambda_2$, we have

$$\begin{aligned} 0 &< \sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p}}{(u(m))^{p(\lambda_1-1)+1}} a_m^p < \infty, \\ 0 &< \int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\lambda_2)-n} g^q(y) dy < \infty, \end{aligned}$$

and the following inequality:

$$\begin{aligned} &\int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{(u'(m))^i}{(u(m)+||y||_\beta^\alpha)^{\lambda+i}} A_m^{(i)} g(y) dy \\ &> \frac{1}{\lambda^i} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1, \lambda_2) \\ &\times \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p} a_m^p}{(u(m))^{p(\lambda_1-1)+1}} \right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\lambda_2)-n} g^q(y) dy \right]^{\frac{1}{q}}. \quad (17) \end{aligned}$$

Proof. By the following expression of the Gamma function:

$$\frac{1}{(u(m)+||y||_\beta^\alpha)^{\lambda+i}} = \frac{1}{\Gamma(\lambda+i)} \int_0^\infty t^{\lambda+i-1} e^{-(u(m)+||y||_\beta^\alpha)t} dt,$$

(13) and Lebesgue term by term theorem (cf. [29]), we have

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda+i)} \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} (u'(m))^i A_m^{(i)} g(y) \left[\int_0^\infty t^{\lambda+i-1} e^{-(u(m)+||y||_\beta^\alpha)t} dt \right] dy \\ &= \frac{1}{\Gamma(\lambda+i)} \int_0^\infty t^{\lambda+i-1} \left(\sum_{m=m_0}^{\infty} e^{-xu(m)} (u'(m))^i A_m^{(i)} \right) \\ &\times \left(\int_{\mathbf{R}_+^n} e^{-||y||_\beta^\alpha t} g(y) dy \right) dt \\ &\geq \frac{1}{\Gamma(\lambda+i)} \int_0^\infty t^{\lambda+i-1} \left(t^{-i} \sum_{m=m_0}^{\infty} e^{-xu(m)} a_m \right) \left(\int_{\mathbf{R}_+^n} e^{-||y||_\beta^\alpha t} g(y) dy \right) dt \\ &= \frac{1}{\Gamma(\lambda+i)} \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} a_m g(y) \left[\int_0^\infty t^{\lambda-1} e^{-(u(m)+||y||_\beta^\alpha)t} dt \right] dy \\ &= \frac{\Gamma(\lambda)}{\Gamma(\lambda+i)} \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{a_m g(y)}{(u(m)+||y||_\beta^\alpha)^\lambda} dy = \frac{I_\lambda}{\lambda^i}. \end{aligned}$$

Then by (15), we have (16). For $\lambda = \lambda_1 + \lambda_2$ in (16), we have (17).

This proves the theorem. □

Theorem 2. With regards to the assumption H1, if $i \in \{0, 1\}$, $\lambda_1 + \lambda_2 = \lambda$, then the constant factor

$$\frac{1}{\lambda^i} \left(\frac{\Gamma^n(\frac{1}{\beta}) B(\lambda - \lambda_2, \lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in (16) is the best value.

Proof. We need to prove that the constant factor in (17) is the best value for $i \in \{0,1\}$. For any $0 < \varepsilon < \min\{|p|\lambda_2, |p|(1-\eta_0), q\lambda_2\}$, we set

$$\begin{aligned}
\widetilde{A}_m^{(0)} &= \widetilde{a}_m := (u(m))^{\lambda_1 - \frac{\varepsilon}{p} - 1} u'(m), \\
\widetilde{A}_m^{(1)} &= \sum_{k=m_0}^{m} \widetilde{a}_k = \sum_{k=m_0}^{m} (u(k))^{(\lambda_1 - \frac{\varepsilon}{p}) - 1} u'(k), m \in \mathbf{N}_{m_0}, \\
\widetilde{g}(y) &: = \begin{cases} 0, ||y||_\beta < 1 \\ ||y||_\beta^{\alpha(\lambda_2 - \frac{\varepsilon}{q}) - n}, ||y||_\beta \geq 1 \end{cases}.
\end{aligned}$$

Since $\varepsilon < |p|(1-\eta_0)$, both

$$(u(x))^{(\lambda_1 - \frac{\varepsilon}{p}) - 1} u'(x) (= (u(x))^{\frac{\varepsilon}{|p|} + \eta_0 - 1} [(u(x))^{\lambda_1 - \eta_0} u'(x)])$$

and $(u(x))^{-\varepsilon - 1} u'(x)$ $(= (u(x))^{-\lambda_1 - \varepsilon + (\eta_0 - 1)} [(u(x))^{\lambda_1 - \eta_0} u'(x)])$ are strictly decreasing in $(m_0 - 1, \infty)$, we find

$$\widetilde{A}_m^{(1)} < \int_{m_0 - 1}^{m} (u(x))^{(\lambda_1 - \frac{\varepsilon}{p}) - 1} u'(x) dx \leq \frac{1}{\lambda_1 - \frac{\varepsilon}{p}} (u(m))^{\lambda_1 - \frac{\varepsilon}{p}},$$

and then for $i \in \{0,1\}$, it follows that

$$\begin{aligned}
\widetilde{A}_m^{(i)} &\leq \frac{(u'(m))^{1-i}}{(\lambda_1 - \frac{\varepsilon}{p})^i} (u(m))^{\lambda_1 - \frac{\varepsilon}{p} + i - 1} \ (m \in \mathbf{N}_{m_0}), \text{ and} \\
& \quad \sum_{m=m_0}^{\infty} \frac{u'(m)}{(u(m))^{\varepsilon+1}} = \frac{u'(m_0)}{(u(m_0))^{\varepsilon+1}} + \sum_{m=m_0+1}^{\infty} \frac{u'(m)}{(u(m))^{\varepsilon+1}} \\
&< \frac{u'(m_0)}{(u(m_0))^{\varepsilon+1}} + \int_{m_0}^{\infty} \frac{u'(x)}{(u(x))^{\varepsilon+1}} dx \\
&= b + \frac{1}{\varepsilon (u(m_0))^{\varepsilon}} \ (b := \frac{u'(m_0)}{(u(m_0))^{\varepsilon+1}}).
\end{aligned}$$

If there exists a positive constant M, with

$$M \geq \frac{1}{\lambda^i} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1, \lambda_2),$$

such that (17) is valid when we replace the constant factor by M, then in particular, by (8) (for $c = \varepsilon$), we have

$$\begin{aligned}
\widetilde{I} &: = \int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{(u'(m))^i \widetilde{A}_m^{(i)} \widetilde{g}(y)}{(u(m) + ||y||_\beta^\alpha)^{\lambda + i}} dy \\
&> M \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p} \widetilde{a}_m^p}{(u(m))^{p(\lambda_1 - 1) + 1}} \right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n - \alpha\lambda_2) - n} \widetilde{g}^q(y) dy \right]^{\frac{1}{q}} \\
&= M \left(\sum_{m=m_0}^{\infty} \frac{u'(m)}{(u(m))^{\varepsilon+1}} \right)^{\frac{1}{p}} \left(\int_{\{y \in \mathbf{R}_+^n; ||y||_\beta \geq 1\}} ||y||_\beta^{-\alpha\varepsilon - n} dy \right)^{\frac{1}{q}} \\
&> \frac{M}{\varepsilon} \left[\varepsilon b + \frac{1}{(u(m_0))^{\varepsilon}} \right]^{\frac{1}{p}} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha \beta^{n-1} \Gamma(\frac{n}{\beta})} \right)^{\frac{1}{q}}.
\end{aligned}$$

By (6), we have

$$\widetilde{I} \leq \frac{1}{(\lambda_1 - \frac{\varepsilon}{p})^i} \sum_{m=m_0}^{\infty} (u(m))^{\lambda_1 - \frac{\varepsilon}{p} + i - 1} u'(m) \int_{y \in \mathbf{R}^n_+;} \frac{||y||_\beta^{\alpha(\lambda_2 - \frac{\varepsilon}{q}) - n} dy}{(u(m) + ||y||_\beta^\alpha)^{\lambda + i}}$$

$$= \frac{}{\beta^{n-1}\Gamma(\frac{n}{\beta})\,(\lambda_1 - \frac{\varepsilon}{p})^{\bar{i}}}$$

$$\times \sum_{m=m_0}^{\infty} (u(m))^{\lambda_1 - \frac{\varepsilon}{p} + i - 1} u'(m) \int_0^\infty \frac{v^{\alpha(\lambda_2 - \frac{\varepsilon}{q}) - 1} dv}{(u(m) + v^\alpha)^{\lambda + i}}$$

$$< \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \frac{1}{(\lambda_1 - \frac{\varepsilon}{p})^i} \sum_{m=m_0}^{\infty} \frac{u'(m)}{(u(m))^{\varepsilon+1}} \int_0^\infty \frac{t^{\lambda_2 - \frac{\varepsilon}{q} - 1} dt}{(1+t)^{\lambda + i}}$$

$$= \frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \frac{B(\lambda_1 + i + \frac{\varepsilon}{q}, \lambda_2 - \frac{\varepsilon}{q})}{(\lambda_1 - \frac{\varepsilon}{p})^i} \left[b + \frac{1}{\varepsilon (u(m_0))^\varepsilon} \right]$$

Based on the above results, we have the following inequality

$$\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \frac{B(\lambda_1 + i + \frac{\varepsilon}{q}, \lambda_2 - \frac{\varepsilon}{q})}{(\lambda_1 - \frac{\varepsilon}{p})^i} \left[\varepsilon b + \frac{1}{(u(m_0))^\varepsilon} \right]$$

$$> \varepsilon \widetilde{I} > M \left[\varepsilon b + \frac{1}{(u(m_0))^\varepsilon} \right]^{\frac{1}{p}} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{q}}$$

For $\varepsilon \to 0^+$, in view of the continuity of the Beta function, we have

$$\frac{B(\lambda_1 + i, \lambda_2)\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})\lambda_1^i} \geq M \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{q}}.$$

Since $i \in (0, 1\}$, $\lambda^i B(\lambda_1 + i, \lambda_2) = \lambda_1^i B(\lambda_1, \lambda_2)$, it follows that

$$\frac{1}{\lambda^i} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1, \lambda_2)$$

$$= \frac{1}{\lambda_1^i} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1 + i, \lambda_2) \geq M.$$

Therefore,

$$M = \frac{1}{\lambda^i} \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1, \lambda_2)$$

is the best value in (17) (namely, for $\lambda_1 + \lambda_2 = \lambda$ in (16)).

This proves the theorem. □

Theorem 3. With regards to the assumption H1, if the constant factor in (16) is the best value, then for $0 \leq \lambda - \lambda_1 - \lambda_2 < |p|\lambda_1$, we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. For $\widehat{\lambda}_1 = \frac{\lambda - \lambda_1 - \lambda_2}{p} + \lambda_1, \widehat{\lambda}_2 = \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$, we find $\widehat{\lambda}_1 + \widehat{\lambda}_2 = \lambda$. For $0 \leq \lambda - \lambda_1 - \lambda_2 < -p\lambda_1$, we observe that $0 < \widehat{\lambda}_1 < \lambda$. and then $0 < \widehat{\lambda}_2 = \lambda - \widehat{\lambda}_1 < \lambda$. Also for $p < 0$,

$$(u(x))^{\widehat{\lambda}_1-\eta_0}u'(x)=(u(x))^{\frac{\lambda-\lambda_1-\lambda_2}{p}}[(u(x))^{\lambda_1-\eta_0}u'(x)]$$

is decreasing in (m_0-1,∞).

By the reverse Hölder's inequality (cf. [28]), we obtain

$$\begin{aligned} B(\widehat{\lambda}_1,\widehat{\lambda}_2) &= \int_0^\infty \frac{u^{\widehat{\lambda}_1-1}}{(1+u)^\lambda}du=\int_0^\infty \frac{(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})}{(1+u)^\lambda}du \\ &\geq \left[\int_0^\infty \frac{u^{\lambda-\lambda_2-1}}{(1+u)^\lambda}du\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda}du\right]^{\frac{1}{q}} \\ &= B^{\frac{1}{p}}(\lambda-\lambda_2,\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1). \end{aligned} \tag{18}$$

Since the constant factor

$$\frac{1}{\lambda^i}\left(\frac{\Gamma^n(\frac{1}{\beta})B(\lambda-\lambda_2,\lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1)$$

in (16) is the best value. compare with the constant factors in (16) and (17) (for $\lambda_1=\widehat{\lambda}_1,\lambda_2=\widehat{\lambda}_2$), we have

$$\begin{aligned} &\frac{1}{\lambda^i}\left(\frac{\Gamma^n(\frac{1}{\beta})B(\lambda-\lambda_2,\lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\ \geq\ &\frac{1}{\lambda^i}\left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}B(\widehat{\lambda}_1,\widehat{\lambda}_2), \end{aligned}$$

namely,

$$B(\widehat{\lambda}_1,\widehat{\lambda}_2)\leq B^{\frac{1}{p}}(\lambda-\lambda_2,\lambda_2)B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1).$$

Hence, (18) keeps the form of equality. The necessary and sufficient condition for taking an equal sign is that there exist constants A and B, such that they are not both zero, and (cf. [28]) $Au^{\lambda-\lambda_2-1}=Bu^{\lambda_1-1}$ a.e. in $\mathbf{R}_+$. Assuming that $A\neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1}=\frac{B}{A}$ a.e. in $\mathbf{R}_+$. It follows that $\lambda-\lambda_2-\lambda_1=0$, and then $\lambda_1+\lambda_2=\lambda$.

This proves the theorem. □

Remark 3. For $\gamma\in(0,1],\lambda_1\in(0,\frac{1}{\gamma})\cap(0,\lambda)$, $\eta_0\in[\lambda_1-\frac{1}{\gamma}+1,1)$, in view of Remark 1, both $u_1(x)=x^\gamma$, $(x\in(0,\infty);m_0=1)$ and $u_2(x)=\ln^\gamma x$ $(x\in(1,\infty);m_0=2)$ satisfy for using Theorem 1-3.

Corollary 1. For $i=0$ in (16), we have the following reverse inequality:

$$\begin{aligned} I &:= \int_{\mathbf{R}_+^n}\sum_{m=m_0}^\infty \frac{a_m g(y)}{(u(m)+||y||_\beta^\alpha)^\lambda}dy \\ &> \left(\frac{\Gamma^n(\frac{1}{\beta})B(\lambda-\lambda_2,\lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})}\right)^{\frac{1}{p}}B^{\frac{1}{q}}(\lambda_1,\lambda-\lambda_1) \\ &\times\left[\sum_{m=m_0}^\infty\frac{(u'(m))^{1-p}a_m^p}{(u(m))^{p(\widehat{\lambda}_1-1)+1}}\right]^{\frac{1}{p}}\left[\int_{\mathbf{R}_+^n}||y||_\beta^{q(n-\alpha\widehat{\lambda}_2)-n}g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{19}$$

In particular, for $\lambda_1+\lambda_2=\lambda$, we have

$$\int_{\mathbf{R}_+^n} \sum_{m=m_0}^{\infty} \frac{a_m g(y)}{(u(m)+||y||_\beta^\alpha)^\lambda} dy > \left(\frac{\Gamma^n(\frac{1}{\beta})}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B(\lambda_1, \lambda_2)$$

$$\times \left[\sum_{m=m_0}^{\infty} \frac{(u'(m))^{1-p} a_m^p}{(u(m))^{p(\lambda_1-1)+1}} \right]^{\frac{1}{p}} \left[\int_{\mathbf{R}_+^n} ||y||_\beta^{q(n-\alpha\lambda_2)-n} g^q(y) dy \right]^{\frac{1}{q}}. \qquad (20)$$

Corollary 2. If $\lambda_1 + \lambda_2 = \lambda$, then the constant factor

$$\left(\frac{\Gamma^n(\frac{1}{\beta}) B(\lambda - \lambda_2, \lambda_2)}{\alpha\beta^{n-1}\Gamma(\frac{n}{\beta})} \right)^{\frac{1}{p}} B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1)$$

in (19) is the best value. On the other hand, if the same constant factor in (19) is the best value, then for $0 \leq \lambda - \lambda_1 - \lambda_2 < |p|\lambda_1$, we have $\lambda_1 + \lambda_2 = \lambda$.

Remark 4. Inequality (16) (resp. (17)) is an extended application of 19) (resp. (20)).

Conclusion

In this paper, following the way of [21] and [22], by means of the weight functions, the idea of introduced parameters, the techniques of real analysis and Abel's partial summation formula, a multidimensional half-discrete reverse Hardy-Hilbert's inequality with the new kernel as

$$\frac{1}{\left(u(m) + \|y\|^{\frac{\alpha}{\beta}\lambda + i}\right)} \; (\alpha, \beta, \lambda > 0, \; i \in \{0, 1\})$$

with one partial sum is obtained in Theorem 1. The equivalent statements of the best value related to several parameters in the new inequality are given in Theorem 2 and Theorem 3. Some particular results are deduced in Corollary 1 and Corollary 2.

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