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A Short Proof and Refinements of Minkowski's Inequality

Youla Yang*

School of Integrated Circuits, Tsinghua University, Beijing 100084, P.R. China

*Corresponding Author

Youla Yang, School of Integrated Circuits, Tsinghua University, Beijing 100084, P.R. China, E-mail: yangyoula111@qq.com

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Abstract

In this paper, inspired by the work of [4,7] and [8-10], we are gonging to present a short proof of the well-known Minkowski's inequality and give an interpolation and a refinement of it.

Keywords: Minkowski's Inequality; Convex Functions; Refinement; Interpolation

Introduction

In [7], the author used an elementary method gave a short proof of the well-known Holder's inequality:

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_p^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}, \quad (1)$$

where all $a_k, b_k > 0$ and $\frac{1}{p} + \frac{1}{q} = 0$ with $p, q > 1$. The equality holds when

$$\frac{a_k^p}{b_k^q} = \frac{a_j^p}{b_j^q},$$

for all $k, j = 1, 2, \cdots, n$. Moreover, if $p = q = 2$, inequality (1) reduces to the well-known Cauchy's inequality:

$$\sum_{i=1}^{n} a_i b_i \leqslant \prod_{i=1}^{n} (a_i^2 + b_i^2)^{\frac{1}{2}}.$$

In [4, 8-10], the authors considered the following function

$$h(t) = \prod_{k=1}^{m} \left[\sum_{i=1}^{n} \left(\prod_{j=1}^{m} a_{ij}\right)^{1-t} (a_{ik}^{p_k})^t\right]^{\frac{1}{p_k}}, \quad (2)$$

and proved that for $0 \leq t_1 < t_2 < \cdots < t_k \leq 1$, then the following interpolation and refinement of the Holder's inequality

$$h(0) = \sum_{i=1}^{n} \prod_{j=1}^{m}, a_{ij} \leqslant h(t_1) \leqslant h(t_2) \leqslant \ldots \leqslant h(t_k) \leqslant h(1) \leqslant \prod_{j=1}^{m} \left(\sum_{i-1}^{m} a_{ij}^{pj}\right)^{\frac{1}{pj}} .(3)$$

Like Holder's inequality, the well-known Minkowski's inequality plays also an important role in the mathematical and physics research fields and literatures. There are the most important, interesting, useful and elementary inequalities in mathematics, physics and other research fields. It plays an important role in mathematics and physics research fields and has great potential in the future research. There were many research papers devoted to the generalizations, refinements and applications of these two important inequalities. For examples, we refer to the references in [1-11] and the references cited in them. Since their importance and application potential both in theory and practical applications, in this paper, inspired by the works of [4,7-10], we are going to present a short proof of the well-known Minkowski's inequality and give a refinement and an interpolation of it. Our results are new and given below.

Theorem 1. If $a_i, b_i > 0, i = 1, 2, \cdots, n, p > 0$, then for $p \geq 1$, we have

$$\left[\sum_{i=1}^{n} (a_i + b_i)^p\right]^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_k^p\right)^{\frac{1}{p}}, \quad (4)$$

and for $0 < p \le 1$, we have

$$\left[\sum_{i=1}^{n}(a_i+b_i)^{p}\right]^{\frac{1}{p}} \geqslant \left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}b_k^p\right)^{\frac{1}{p}}. \quad (5)$$

Theorem 2. Define a C^{∞} function $g(x)$ as follows

$$g(x) = \left[\sum_{i=1}^{n}(a_k+b_k)^p\right]^{\frac{1-x}{p}} \left[\left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}b_k^p\right)^{\frac{1}{p}}\right]^{x}. \quad (6)$$

Then for $p = 1$, $g(x) \equiv \sum_{k=1}^{n}(a_k+b_k) \equiv g(0) = constant.$

For $p > 1$, $g'(x) \ge 0$ and for $0 \le x_1 < x_2 < \cdots < x_m \le 1$, the following inequalities are refinements and interpolation of $g(x)$:

$$\left[\sum_{i=1}^{n}(a_k+b_k)^p\right]^{\frac{1}{p}} = g(0) \leqslant g(x_1) \leqslant (x_2)... \leqslant g(x_m) \leqslant g(1) = \left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}b_k^p\right)^{\frac{1}{p}}.(7)$$

For $0 < p < 1$, $g'(x) \le 0$ and for $0 \le x_1 < x_2 < \cdots < x_m \le 1$, the following inequalities are refinements and interpolations of $g(x)$.

$$\left[\sum_{i=1}^{n}(a_k+b_k)^p\right]^{\frac{1}{p}} = g(0) \geqslant g(x_1) \geqslant (x_2)... \geqslant g(x_m) \geqslant g(1) = \left(\sum_{k=1}^{n}a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}b_k^p\right)^{\frac{1}{p}}.(8)$$

Moreover,

$$g'(x) = g(x) In\left[\frac{g(1)}{g(0)}\right], g''(x) = g(x)\left(In\left[\frac{g(1)}{g(0)}\right]\right)^2 \geqslant 0, \forall x \in [0,1],$$

and $g''(x) \equiv 0$ if and only if $g(1) = g(0)$, in this case, $g'(x) \equiv 0$ and $g(x) \equiv g(0) =$ constant.

Proof of Theorem 1: For $1 \le m \le n$, set $x = a_m$ and define

$$f_m(x) = (x^p + A_m)^{\frac{1}{p}} + B - [(x+b_m)^p + C_m]^{\frac{1}{p}} \quad (9)$$

Where

$$A_m = \sum_{k \ne m} a_k^p, B + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}, C_m = \sum_{k \ne m}(a_k+b_k)^p. \quad (10)$$

Hence

$$f_m'(x) = (x^p + A_m)^{\frac{1-p}{p}} x^{p-1} - [(x + b_m) + C_m]^{\frac{1-p}{p}} (x + b_m)^{p-1} = \frac{x^{p-1}}{(x^p + A_m)^{\frac{p-1}{p}}} - \frac{(x + b_m)^{p-1}}{[(x + b_m)^p + C_m]^{\frac{p-1}{p}}}.$$

Solving equation $f(\overline{x}) = 0$, we get

$$\frac{\overline{x}^{p-1}}{(\overline{x}^p + A_m)^{\frac{p-1}{p}}} = \frac{(\overline{x} + b_m)^{p-1}}{[(\overline{x} + b_m)^p + C_m]^{\frac{p-1}{p}}}$$

and after some calculations, we obtain

$$\overline{x} = \frac{b_m A_m^{\frac{1}{p}}}{C_m^{\frac{1}{p}} - A_m^{\frac{1}{p}}}, \quad (11)$$

$$\frac{\overline{x}}{(\overline{x}^p + A_m)^{\frac{1}{p}}} = \frac{\overline{x} + b_m}{[(\overline{x} + b_m)^p + C_m]^{\frac{1}{p}}}. \quad (12)$$

It follows from equation (9) and $\overline{x} = a_m$ that

$$\frac{a_m + b_m}{a_m} = \frac{\left[\sum_{k=1}^n (a_k + b_k)^p\right]^{\frac{1}{p}}}{\left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}}} = c_0(> 1) = constant. \quad (13)$$

That is $\frac{b_m}{a_m} = \lambda = c_0 - 1(> 0) = constant$. Let $m = 1, 2, \cdots, n$, we get $b_k = \lambda_{a_k}$, $k = 1, 2, \cdots, n$. Substituting above equations into (6), we get

$$f_m(\overline{x}) = \left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n b_k^p\right)^{\frac{1}{p}} - \left[\sum_{k=1}^n (a_k + b_k)^p\right]^{\frac{1}{p}}$$

$$\left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} + \lambda\left(\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}} - (1+\lambda)\left[\sum_{k=1}^n a_k^p\right]^{\frac{1}{p}}$$

$$(1+\lambda)\left[\sum_{k=1}^n a_k^p\right]^{\frac{1}{p}} - (1+\lambda)\left[\sum_{k=1}^n a_k^p\right]^{\frac{1}{p}}$$

$$= 0.$$

Since

$$f_m''(x) = (p-1)\left\{A_m x^{p-2}(x^p + A_m)^{\frac{1}{p}-2} - C_m\left[(x+b_m)^p + C_m\right]^{\frac{1}{p}-2}(x+b_m)^{p-2}\right\},$$

we get from (8), (9) that

$$f_m''(x) = \frac{(p-1)A^{1+\frac{1}{p}} b_m \bar{x}^{p-3}(\bar{x}^p + A_m)^{\frac{1}{p}-2}}{C_m^{\frac{1}{p}}}.$$

If $p = 1$, then $f_m(x) \equiv 0$, in this case, (4) and (5) becomes equality for all $a_k \geq 0$, $b_k \geq 0$, $1 \leq k \leq n$.

If $p > 1$, then $f_m(\bar{x}) > 0$ for all $\bar{x} > 0$, hence $f_m(\bar{x})$ achieves its minimum at $x = \bar{x}$ where $\bar{x}$ satisfies $f_m(x) = 0$, in this case inequality (4) holds.

If $0 < p < 1$, then $f_m(\bar{x}) < 0$ for all $\bar{x} > 0$, hence $f_m(x)$ achieves its maximum at $x = \bar{x}$ where $\bar{x}$ satisfies $f_m(\bar{x}) = 0$, in this case inequality (5) holds. Theorem 1 is proved.

Proof of Theorem 2: It follows from the expression of $g(x)$ that

$$g(x) = g(0)\left[\frac{g(1)}{g(0)}\right]^x, g'(x) = g(x)In\left[\frac{g(1)}{g(0)}\right], g''(x) = g(x)\left(\left[\left[\frac{g(1)}{g(0)}\right]\right]\right)^2 \geqslant 0.$$

By Theorem 1, if $p = 1$, then $g(x) \equiv g(0) = \Sigma_{k=1}^{n}(a_k + b_k) = g(1)$.

If $p > 1$, then $g(1) \geq g(0)$, with equality holds if and only if bk/ak = constant, $k = 1, 2, \cdots, n$. In this case,

$$g'(x) = g(x)\left[\frac{g(1)}{g(0)}\right] \geqslant 0, g''(x) = g(x)\left(In\left[\frac{g(1)}{g(0)}\right]\right)^2 \geqslant 0.$$

Then (7) is an interpolation and a refinement of (4).

If $0 < p < 1$, then $g(1) \leq g(0)$, with equality holds if and only if bk/ak = constant, $k = 1, 2, \cdots, n$. In this case,

$$g'(x) = g(x)In\left[\frac{g(1)}{g(0)}\right] \leqslant 0, g''(x) = g(x)\left(In\left[\frac{g(1)}{g(0)}\right]\right)^2 \geqslant 0.$$

Then (8) is an interpolation and a refinement of (5). Theorem 2 is proved.

Declarations

Ethical Approval

Consent to Publish.

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Data availability

No data was used for the research described in the article.

Conflict of Interest

Not Applicable.

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