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Solving Cauchy's Problem in the 2D Fractional Diffraction Crystal Microtomography

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Abstract

The Cauchy problem of the 2D fractional X-ray diffraction optics designed to describe the mathematical model of the X-ray propagation via the imperfect crystal has been described in terms of the matrix integral Fredholm–Volterra equation. Using the general Green function formalism, the matrix Resolvent solution of the Cauchy problem of the 2D fractional X-ray diffraction optics has been built and analyzed for the case of the coherent two-beam X-ray diffraction by imperfect crystals under the non-locality interaction of the X-ray with atoms of crystal medium along the crystal thickness. In the case, when the crystal-lattice elastic displacement field is the linear function $f(\boldsymbol{R}) = ax + b$, coefficients $a, b = const$, the analytical solution of the 2D fractional diffraction optics Cauchy problem has been obtained and analysed for arbitrary fractional order parameter α, $\alpha \in (0, 1]$.

Keywords: Diffraction Optics System of Fractional Differential Equations; The Gerasimov–Caputo Differential Operator; The Cauchy Problem; Matrix Fredholm– Volterra Integral Equation of the Second Kind

1 Introduction

Normally in literature, the 2D X-ray diffraction optics theory has been based on the dif- ferential partial-in-derivative Takagi–Taupin (TT) equations when the fractional-order parameter $\alpha = 1$ (see [1]). In the last decades, substantial progress has been achieved in mathematical physics using differential equations with fractional order derivatives [2]. Cauchy problems for systems of fractional differential equations, which act as a mathematics basis for various physical models have been studied [3–7].

Following this logic, one can push one step further in the diffraction optics theory now founded on the TT-type equations with the fractional derivatives of the arbitrary order $\alpha \in (0, 1]$ along the direction 0*z* of the energy flow propagation within a crystal medium. In this paper, based on the technique of double Fourier–Laplace transform, the integral matrix Fredholm–Volterra equation of the second kind is derived, which is equivalent to the two- dimensional Cauchy problem of diffractive optics. The work goal is to develop the integral formalism of the two-dimensional theory of diffractive optics, previously proposed by the authors [10], based on the fractional Cauchy problem. In the case, when the imperfect crystal displacement field function $f(\boldsymbol{R})$ is a linear function of x, namely: $f(\boldsymbol{R}) = ax + b$, and $a, b = const$, one finds out an analytical solution of the Cauchy problem for an arbitrary fractional-order parameter (FOP) α, $\alpha \in (0, 1]$.
Accordingly, the original system of fractional diffraction optics equations takes the form (*cf.* [7])

$$\begin{pmatrix} \partial_{0t}^{\alpha} - \frac{\partial}{\partial x} & 0 \\ 0 & \partial_{0t}^{\alpha} + \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} E_0(x,t) \\ E_h(x,t) \end{pmatrix} =$$

$$= i \begin{pmatrix} \gamma & \sigma \exp[if(x,t)] \\ \sigma \exp[-if(x,t)] & \gamma \end{pmatrix} \begin{pmatrix} E_0(x,t) \\ E_h(x,t) \end{pmatrix}, \tag{1}$$

with the Cauchy problem's condition

$$\begin{pmatrix} E_0(x,0) \\ E_h(x,0) \end{pmatrix} = \begin{pmatrix} \varphi_0(x) \\ \varphi_h(x) \end{pmatrix}, \quad -\infty < x < \infty, \tag{2}$$

where

$$\partial_{at}^{\nu} g(t) = \operatorname{sgn}^n(t-a) D_{at}^{\nu-n} \frac{d^n}{dt^n} g(t), \quad n-1 < \nu \le n, \quad n \in \mathbb{N}, \tag{3}$$

is the Gerasimov–Caputo fractional derivative beginning at point a (cf. [3]), D_{ay}^{ν} is the Riemann–Liouville fractional integro-differential operator of order ν is equal to

$$D_{ay}^{\nu} g(y) = \frac{\operatorname{sgn}(y-a)}{\Gamma(-\nu)} \int_a^y \frac{g(s)ds}{|y-s|^{\nu+1}}, \quad \nu < 0,$$

and for $\nu \ge 0$ the operator D_{ay}^{ν} can be determined by recursive relation

$$D_{ay}^{\nu} g(y) = \operatorname{sgn}(y-a) \frac{d}{dy} D_{ay}^{\nu-1} g(y), \quad \nu \ge 0, \tag{4}$$

$\Gamma(z)$ is the Euler gamma-function, $\varphi_0(x)$ and $\varphi_h(x)$ are the given real-valued functions.
Note that in the limit case of the FOP $\alpha = 1$ the operator $\partial_{0t}^{\alpha} g(t)$ reduces to the
standard derivative $\frac{d}{dt} g(t)$. dt
The main point of this paper is to solve the boundary-valued Cauchy's problem in the 2D 'fractional' X-ray diffraction crystal optics taking into account the non-locality of the X-ray-crystal medium interaction.
The paper is organized as follows: Section 1 contains an introductory part, in which the purpose and ideology of the work are explained.

In Section 2, using the method of dou- ble Fourier–Laplace integral transform, Cauchy's problem is reduced to the Fredholm– Volterra integral matrix equation of the second kind.

In Section 3, an explicit solution of Cauchy's problem (1) – (2) is obtained in the case of a linear function f(R).

In Section 4, it is shown that in the case of constant initial amplitudes (conditions of Cauchy's problem), this solution is expressed through two-parameter Mittag-Leffler functions. In Section 5, conclusions are presented regarding the prospects of the considered theoretical approach to modeling two-dimensional X-ray diffraction scattering.

2 Reducing Cauchy's problem to the matrix integral Fredholm– Volterra equation

Let us convert the Cauchy problem in the 'differential derivative' form (1)–(2) to the Fredholm– Volterra-type integral matrix equation of the second kind. The construction of the resolvent of this equation in terms of a Liouville-Neumann series is of great importance for computer modeling and subsequent reconstruction of the crystal displacement field function f(R) from X-ray diffraction microtomography data.

The system of differential TT-type equations (1) may be rewritten into the form

$$\begin{pmatrix} O_-^\alpha - i\gamma & 0 \\ 0 & O_+^\alpha - i\gamma \end{pmatrix} \mathbf{E} = i\sigma \mathbf{K}\mathbf{E}, \qquad (5)$$

where

$$O_+^\alpha = \partial_{0t}^\alpha + \frac{\partial}{\partial x}, \quad O_-^\alpha = \partial_{0t}^\alpha - \frac{\partial}{\partial x},$$

$$\mathbf{E} \equiv \mathbf{E}(x,t) = \begin{pmatrix} E_0(x,t) \\ E_h(x,t) \end{pmatrix}, \quad \mathbf{K} \equiv \mathbf{K}(x,t) = \begin{pmatrix} 0 & e^{if(x,t)} \\ e^{-if(x,t)} & 0 \end{pmatrix}.$$

Acting onto both sides of (5) by the operator $diag(O_+^\alpha - i\gamma, O_-^\alpha - i\gamma)$,taking into account that the column vector **E** = **E**(*x*, *t*) is the solution of Eq. (5) and $\mathbf{K}^2$ is equal to Unit matrix, one can obtain

$$\begin{pmatrix} (O_+^\alpha - i\gamma)(O_-^\alpha - i\gamma) + \sigma^2 & 0 \\ 0 & (O_-^\alpha - i\gamma)(O_+^\alpha - i\gamma) + \sigma^2 \end{pmatrix} \mathbf{E} =$$
$$= i\sigma \left(\partial_{0t}^\alpha + if_x'\right)(\mathbf{K}\mathbf{E}) - i\sigma \mathbf{K}\left(\partial_{0t}^\alpha \mathbf{E}\right). \qquad (6)$$

Further, we denote the Fourier transform of the function *f(x)* by $(f(x))_k$, the Laplace transform of the function *g(t)* by $(g(t))_p$, and respectively, the double Fourier–Laplace transform of the function *h(x, t)* by $(h(x,t))_{k,p}$.

Using the following formula for the Laplace transform of the fractional derivative

$$[\partial_{0t}^\alpha H(x,t)]_p = p^\alpha [H(x,t)]_p - p^{\alpha-1} H(x,0),$$

one can get

$$[O_\pm^\alpha H(x,t)]_{k,p} = (p^\alpha \pm ik)[H(x,t)]_{k,p} - p^{\alpha-1}[H(x,0)]_k, \qquad (7)$$

Keeping in mind Eq. (7), and applying the double Fourier–Laplace transform to Eq. (6), one obtains

$$\begin{pmatrix} E_0(x,t) \\ E_h(x,t) \end{pmatrix}_{k,p} = \frac{p^{\alpha-1}}{(p^\alpha - i\gamma)^2 + k^2 + \sigma^2} \times$$

$$\times \left\{ \begin{pmatrix} p^\alpha - i\gamma + ik & 0 \\ 0 & p^\alpha - i\gamma - ik \end{pmatrix} \begin{pmatrix} E_0(x,0) \\ E_h(x,0) \end{pmatrix}_k + i\sigma \begin{pmatrix} e^{if(x,0)} E_h(x,0) \\ e^{-if(x,0)} E_0(x,0) \end{pmatrix}_k \right\} +$$

$$+ \frac{1}{(p^\alpha - i\gamma)^2 + k^2 + \sigma^2} \left\{ \begin{pmatrix} \partial_{0t}^\alpha + if'_x & 0 \\ 0 & \partial_{0t}^\alpha + if'_x \end{pmatrix} \begin{pmatrix} i\sigma e^{if} E_h(x,t) \\ i\sigma e^{-if} E_0(x,t) \end{pmatrix} - \right.$$

$$\left. - \begin{pmatrix} 0 & i\sigma e^{if} \\ i\sigma e^{-if} & 0 \end{pmatrix} \begin{pmatrix} \partial_{0t}^\alpha E_0(x,t) \\ \partial_{0t}^\alpha E_h(x,t) \end{pmatrix} \right\}_{k,p}. \quad (8)$$

Applying the Efros theorem for operational calculus, the formula for the Laplace trans- form of the Wright function [2, 3]

$$\left(y^{\delta-1} \phi(-\beta, \mu; -ty^{-\beta}) \right)_p = p^{-\mu} e^{-p^\beta t}, \quad (9)$$

the following well-known integrals

$$\int_0^\infty \frac{\cos kx}{k^2 + \rho^2} dk = \frac{\pi}{2\rho} e^{-\rho x}, \quad (10)$$

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-x\sqrt{p^2+\sigma^2}} e^{pt} dp}{\sqrt{p^2+\sigma^2}} = J_0 \left(\sigma \sqrt{t^2 - x^2} \right) \Theta(t - |x|), \quad (11)$$

the inverse Fourier–Laplace transform for the relation (8) takes the form

$$\mathbf{E}(x,t) = (\mathbf{A}^{\alpha,\gamma} \mathbf{E}(x,t))(x,t) + (\mathbf{B}^{\alpha,\gamma} \mathbf{E}(x,0))(x,t), \quad (12)$$

where

$$(\mathbf{A}^{\alpha,\gamma} \mathbf{E}(x,t))(x,t) = -i\sigma \int_0^t dv \int_{-\infty}^{\infty} \mathbf{K}_1(x,t;u,v) \mathbf{E}(u,v) du -$$

$$-i\sigma \int_{-\infty}^{\infty} D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x-u,t) \mathbf{K}(u,0) \mathbf{E}(u,0) du, \quad (13)$$

$$(\mathbf{B}^{\alpha,\gamma} \mathbf{E}(x,0))(x,t) = -\int_{-\infty}^{\infty} D_{0t}^{2\alpha-1} G_{\alpha,\gamma}(x-u,t) \mathbf{E}(u,0) du -$$

$$-i\gamma \int_{-\infty}^{\infty} D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x-u,t) \mathbf{E}(u,0) du + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \int_{-\infty}^{\infty} D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x-u,t) \mathbf{E}'(u,0) du +$$

$$+i\sigma \int_{-\infty}^{\infty} D_{0t}^{\alpha-1} G_{\alpha,\gamma}(x-u,t) \mathbf{K}(u,0) \mathbf{E}(u,0) du, \quad (14)$$

$$\mathbf{K}_1(x,t;u,v) = D_{vt}^\alpha G_{\alpha,\gamma}(x-u,t-v) \cdot \mathbf{K}(u,v) +$$

$$+if_u'(u,v)\cdot G_{\alpha,\gamma}(x-u,t-v)\mathbf{K}(u,v)+D_{vt}^{\alpha}\left[G_{\alpha,\gamma}(x-u,t-v)\mathbf{K}(u,v)\right],$$

$\Theta(x)$ is the Heaviside function, $J_0(x)$ is the zero-order Bessel function,

$$\phi(\beta,\rho;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\beta k+\rho)},\quad \beta>-1,\quad \rho\in\mathbb{C}$$

is the Wright function (see, e,g., [3]), and

$$G_{\alpha,\gamma}(x,t)=\frac{1}{2}\int_{|x|}^{\infty}e^{i\gamma\tau}J_0\left(\sigma\sqrt{\tau^2-x^2}\right)\frac{1}{t}\phi\left(-\alpha,0;-\frac{\tau}{t^{\alpha}}\right)d\tau \tag{15}$$

is the Green function introduced in [7].

Taking into account Eqs. (13), (14), the integral matrix equation (12) may be to reduce

$$\mathbf{E}(x,t)+i\sigma\int_0^t dv\int_{-\infty}^{\infty}\mathbf{K}_1(x,t;u,v)\mathbf{E}(u,v)du=\mathbf{F}(x,t), \tag{16}$$

where

$$\mathbf{F}(x,t)=(\mathbf{B}^{\alpha,\gamma}\mathbf{E}(x,0))(x,t)-i\sigma\int_{-\infty}^{\infty}D_{0t}^{\alpha-1}G_{\alpha,\gamma}(x-u,t)\mathbf{K}(u,0)\mathbf{E}(u,0)du.$$

Thus, according to Eq. (16), the Cauchy problem for the matrix Eq. (1) is reduced to the system of Fredholm-Volterra integral equations. The unique solution of the matrix integral equation (16) has the form

$$\mathbf{E}(x,t)=\mathbf{F}(x,t)-i\sigma\int_0^t\int_{-\infty}^{+\infty}\mathbf{R}(x,t;\xi,\eta)\mathbf{F}(\xi,\eta)d\xi d\eta,$$

where

$$\mathbf{R}(x,t;\xi,\eta)=\sum_{n=0}^{\infty}(-i\sigma)^n\mathbf{K}_{n+1}(x,t;\xi,\eta), \tag{17}$$

$$\mathbf{K}_n(x,t;\xi,\eta)=\int_{\eta}^{t}dv\int_{-\infty}^{\infty}\mathbf{K}_{n-1}(x,t;u,v)\mathbf{K}_1(u,v;\xi,\eta)du.$$

The convergence of series (17) can be easily established from the properties of function (15).

3 The Cauchy problem. The crystal-lattice displacement field function *f (R) = ax + b*

Here we will build up a solution of the basic fractional Cauchy problem when the crystal- lattice displacement field function *f (R) = ax + b*.

After trivial exponentional substitutions for the wave amplitudes $E_0(x,t)$, $Eh(x,t)$, the system (1) can written down as (for simplicity, further, the same notations for the wave amplitudes $E_0(x,t)$, $Eh(x,t)$, are to be saved)

$$\begin{aligned} &\left(\partial_{0t}^{\alpha} - \frac{\partial}{\partial x}\right) E_0(x,t) = i\gamma E_0(x,t) + i\sigma e^{iax} E_h(x,t), \\ &\left(\partial_{0t}^{\alpha} + \frac{\partial}{\partial x}\right) E_h(x,t) = i\sigma e^{-iax} E_0(x,t) + i\gamma E_h(x,t). \end{aligned} \tag{18}$$

Substituting the functions $E_0(x,t)$, $E_h(x,t)$ as

$$E_0(x,t) = \exp\left(i\frac{ax}{2}\right)\mathcal{E}_0(x,t), \quad E_h(x,t) = \exp\left(-i\frac{ax}{2}\right)\mathcal{E}_h(x,t),$$

one obtains

$$\begin{aligned} &\left(\partial_{0t}^{\alpha} - \frac{\partial}{\partial x}\right) \mathcal{E}_0(x,t) = i\left(\gamma + \frac{a}{2}\right)\mathcal{E}_0(x,t) + i\sigma \mathcal{E}_h(x,t), \\ &\left(\partial_{0t}^{\alpha} + \frac{\partial}{\partial x}\right) \mathcal{E}_h(x,t) = i\sigma \mathcal{E}_0(x,t) + i\left(\gamma + \frac{a}{2}\right)\mathcal{E}_h(x,t), \end{aligned} \tag{19}$$

and the initial wavefield amplitudes condition

$$\begin{pmatrix} \mathcal{E}_0(x,0) \\ \mathcal{E}_h(x,0) \end{pmatrix} = \begin{pmatrix} \psi_0(x) \\ \psi_h(x) \end{pmatrix} = \begin{pmatrix} e^{-iax/2}\varphi_0(x) \\ e^{iax/2}\varphi_h(x) \end{pmatrix}, \quad -\infty < x < \infty. \tag{20}$$

Then, let us introduce the notations

$$\Gamma(x,t) = \frac{1}{2}\int_{|x|}^{\infty} g(t,\tau)\mathbf{Q}(x,\tau)d\tau + \Gamma_0(x,t), \tag{21}$$

$$\mathbf{Q}(x,\tau) = \begin{bmatrix} -\sigma\frac{\tau-x}{\sqrt{\tau^2-x^2}}J_1(\sigma\sqrt{\tau^2-x^2}) & i\sigma J_0(\sigma\sqrt{\tau^2-x^2}) \\ i\sigma J_0(\sigma\sqrt{\tau^2-x^2}) & -\sigma\frac{\tau+x}{\sqrt{\tau^2-x^2}}J_1(\sigma\sqrt{\tau^2-x^2}) \end{bmatrix}, \tag{22}$$

$$\Gamma_0(x,t) = g(t,|x|)\begin{bmatrix} \Theta(-x) & 0 \\ 0 & \Theta(x) \end{bmatrix}, \tag{23}$$

$$g(t,\tau) = \frac{e^{i(\gamma+a/2)\tau}}{t}\phi\left(-\alpha,0;-\tau t^{-\alpha}\right),$$

$\Theta(x)$ is the Heaviside function, $J_m(z)$ is the m-order Bessel function of the argument z. According to [9], the solution of the Cauchy problem (19) – (20) has the form

$$\mathcal{E}(x,t) = \int_{-\infty}^{\infty} D_{0t}^{\alpha-1}\Gamma(x-\xi,t)\psi(\xi)d\xi, \tag{24}$$

in the class of function

$$\mathcal{E}(x,t) \in C(\overline{\Omega}), \quad \partial_{0t}^{\alpha}\mathcal{E}(x,t), \frac{\partial}{\partial x}\mathcal{E}(x,t) \in C(\Omega),$$

where $\psi(x) = [\psi_0(x), \psi_h(x)]^{tr} \in C(-\infty,\infty)$ is the function, which satisfies the Holder's condition, and the following relation as $|x| \to \infty$

$$\psi(x) = O(\exp(\rho|x|^{\varepsilon})), \quad \varepsilon = \frac{1}{1-\alpha}, \quad \rho < (1-\alpha)(\alpha T^{-1})^{\frac{\alpha}{1-\alpha}}.$$

Underline the solution to the Cauchy problem (19) – (20) is unique in the class of functions, which satisfy the condition

$$\mathcal{E}(x,t) = O(\exp(k|x|^{\varepsilon})), \quad \text{as} \quad |x| \to \infty,$$

for some k > 0.

From Eqs. (20) – (24) it follows the solution of the Cauchy problem (18), (2) can be cast into the form

$$\mathbf{E}(x,t) = \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1}\mathbf{G}(x,\xi,t)\varphi(\xi)d\xi, \tag{25}$$

where the following notations are introduced

$$\mathbf{G}(x,\xi,t) = \frac{1}{2}\int\limits_{|x-\xi|}^{\infty} g(t,\tau)\mathbf{S}(x,\xi,\tau)d\tau + \mathbf{G}_0(x,\xi,t), \tag{26}$$

$$\mathbf{S}(x,\xi,\tau) = \begin{bmatrix} -\sigma e^{iaX_1/2}\frac{\tau-X_1}{\sqrt{\tau^2-X_1^2}}h_1(\tau,X_1) & i\sigma e^{iaX_2/2}h_0(\tau,X_1) \\ i\sigma e^{-iaX_2/2}h_0(\tau,X_1) & -\sigma e^{-iaX_1/2}\frac{\tau+X_1}{\sqrt{\tau^2-X_1^2}}h_1(\tau,X_1) \end{bmatrix},$$

$$h_0(\tau,X_1) = J_0(\sigma\sqrt{\tau^2-X_1^2}), \quad h_1(\tau,X_1) = J_1(\sigma\sqrt{\tau^2-X_1^2})$$

$$\mathbf{G}_0(x,\xi,t) = \begin{bmatrix} e^{-i\gamma X_1}\Theta(-X_1) & 0 \\ 0 & e^{i\gamma X_1}\Theta(X_1) \end{bmatrix} \frac{1}{t}\phi\left(-\alpha, 0; -|X_1|t^{-\alpha}\right), \tag{27}$$

$$X_1 = x-\xi, \quad X_2 = x+\xi.$$

4 Case of the function f(R) = ax + b and E0(x,0) = 1, Eh(x, 0) = 0

Let us consider the case when the initial conditions for the wavefield amplitudes *E0(x,0)* and *Eh(x,0)* are constant and equal to

$$E_0(x,0) = \varphi_0(x) \equiv 1, \quad E_h(x,0) = \varphi_h(x) \equiv 0. \tag{28}$$

Then, the formulae (25)–(27) can be simplified and expressed in terms of the Mittag– Leffler-type functions. Keeping in mind Eq. (28), Eq. (25) for ***E****(x, t)* can be written down as

$$\mathbf{E}(x,t) = \int\limits_{-\infty}^{\infty} D_{0t}^{\alpha-1}\mathbf{G}(x,\xi,t)\begin{pmatrix}1\\0\end{pmatrix}d\xi =$$

$$= \frac{1}{2}\int\limits_{-\infty}^{\infty}d\xi\int\limits_{|x-\xi|}^{\infty} D_{0t}^{\alpha-1}g(t,\tau)\mathbf{S}(x,\xi,\tau)\begin{pmatrix}1\\0\end{pmatrix}d\tau +$$

$$+\int_{-\infty}^{\infty} D_{0t}^{\alpha-1}\mathbf{G}_0(x,\xi,t)\begin{pmatrix}1\\0\end{pmatrix}d\xi \equiv \mathbf{I}_1(x,t)+\mathbf{I}_2(x,t). \tag{29}$$

By changing the integration order, let us evaluate the integral $\mathbf{I}_1(x, t)$.

$$\mathbf{I}_1(x,t)=\frac{1}{2}\int_0^{\infty} D_{0t}^{\alpha-1}g(t,\tau)d\tau\int_{x-\tau}^{x+\tau}\mathbf{S}(x,\xi,\tau)\begin{pmatrix}1\\0\end{pmatrix}d\xi =$$

$$=\frac{1}{2}\int_0^{\infty} D_{0t}^{\alpha-1}g(t,\tau)d\tau\int_{-\tau}^{\tau}\mathbf{S}(x,x-\eta,\tau)\begin{pmatrix}1\\0\end{pmatrix}d\eta. \tag{30}$$

Let's calculate the integrals involved in the representation of the term $\mathbf{I}_1(x, t)$. We will need

$$\int_0^{\tau}\frac{\eta}{\sqrt{\tau^2-\eta^2}}\cos\left(\rho\sqrt{\tau^2-\eta^2}\right)J_0(\sigma\eta)\,d\eta=\frac{1}{k}\sin(k\tau), \tag{31}$$

$$\int_0^{\tau}\frac{J_1(\sigma\eta)}{\sqrt{\tau^2-\eta^2}}\cos\left(\rho\sqrt{\tau^2-\eta^2}\right)d\eta=\frac{1}{\sigma\tau}\cos(\rho\tau)-\frac{1}{\sigma\tau}\cos(k\tau), \tag{32}$$

where $k=\sqrt{\sigma^2+\rho^2}$.

Using formula (32), one can obtain

$$\int_0^{\tau}\sin\left(\rho\sqrt{\tau^2-\eta^2}\right)J_1(\sigma\eta)\,d\eta=\frac{1}{\sigma}\sin(\rho\tau)-\frac{\rho}{\sigma k}\sin(k\tau). \tag{33}$$

From the Eqs. (31)–(33), it directly follows up

$$\int_{-\tau}^{\tau}S_{11,22}(x,x-\eta,\tau)d\eta=-\sigma\int_{-\tau}^{\tau}e^{\pm ia\eta/2}\frac{\tau\mp\eta}{\sqrt{\tau^2-\eta^2}}J_1(\sigma\sqrt{\tau^2-\eta^2})d\eta=$$

$$=2\cos(k\tau)-i\frac{a}{k}\sin(k\tau)-2e^{-ia\tau/2}, \tag{34}$$

$$\int_{-\tau}^{\tau}S_{12,21}(x,x-\eta,\tau)d\eta=i\sigma e^{\pm iax}\int_{-\tau}^{\tau}e^{\mp ia\eta/2}J_0(\sigma\sqrt{\tau^2-\eta^2})d\eta=$$

$$=i\sigma\frac{2}{k}e^{\pm iax}\sin(k\tau), \tag{35}$$

where Sij $(x, x-\eta, \tau)$ $(i, j = 1, 2)$ are the elements of matrix $\mathbf{S}(x, x-\eta, \tau)$; and

$$k=\sqrt{a^2/4+\sigma^2}.$$

Next, after some routine calculations, from (30), (34) and (35) one obtains

$$\mathbf{I}_1(x,t)=\int_0^\infty t^{-\alpha}\phi\left(-\alpha,1-\alpha;-\tau t^{-\alpha}\right)\mathbf{N}(x,\tau)\begin{pmatrix}1\\0\end{pmatrix}d\tau, \tag{36}$$

where

$$\mathbf{N}(x,\tau)=\mathbf{N}_1(x)e^{i(a_1+k)\tau}+\mathbf{N}_2(x)e^{i(a_1-k)\tau}-\begin{pmatrix}1&0\\0&1\end{pmatrix}e^{i\gamma\tau},$$

$$\mathbf{N}_{1,2}(x)=\frac{1}{2}\begin{pmatrix}1\mp\frac{a}{2k} & \pm\frac{\sigma}{k}e^{iax}\\ \pm\frac{\sigma}{k}e^{-iax} & 1\mp\frac{a}{2k}\end{pmatrix},$$

where $a_1=\gamma+\frac{a}{2}$.

From (27) one finds out

$$\mathbf{I}_2(x,t)=\int_0^\infty t^{-\alpha}\phi\left(-\alpha,1-\alpha;-\eta t^{-\alpha}\right)e^{i\gamma\eta}\begin{pmatrix}1\\0\end{pmatrix}d\eta. \tag{37}$$

It is known that following Stankovic's transformation integral (see [2,3])

$$\int_0^\infty \exp(\lambda\tau)t^{\nu-1}\phi\left(-\mu,\nu;-\tau t^{-\mu}\right)d\tau=t^{\mu+\nu-1}E_{\mu,\mu+\nu}\left(-\lambda t^{\mu}\right), \tag{38}$$

takes place for any $\lambda\in\mathbb{C}$, $\mu\in(0,1)$, $\nu\in\mathbb{R}$,
where

$$E_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+\rho k)}$$

is the Mittag–Leffler-type function [8].

Applying formula (38) to equalities (36) and (37), the total solution (29) can be cast into the form

$$\mathbf{E}(x,t)=\frac{1}{2}\begin{pmatrix}1-\frac{a}{2k} & 1+\frac{a}{2k}\\ \frac{\sigma}{k}e^{-iax} & -\frac{\sigma}{k}e^{-iax}\end{pmatrix}\begin{pmatrix}E_{\alpha,1}\left(i(a_1+k)t^{\alpha}\right)\\ E_{\alpha,1}\left(i(a_1-k)t^{\alpha}\right)\end{pmatrix}. \tag{39}$$

Using the properties of a Mittag-Leffler type function, it is easy to show that function (39) provides the proper solution of Cauchy's problem (18).

In the case when the FOP $\alpha\to 1$, in view of the relation

$$E_{1,1}(z)=\exp(z),$$

the total solution (39) is reduced to (*cf.* [7])

$$\mathbf{E}(x,t)=\frac{1}{2}\begin{pmatrix}1-\frac{a}{2k} & 1+\frac{a}{2k}\\ \frac{\sigma}{k}e^{-iax} & -\frac{\sigma}{k}e^{-iax}\end{pmatrix}\begin{pmatrix}e^{i(a_1+k)t}\\ e^{i(a_1-k)t}\end{pmatrix}. \tag{40}$$

5 Conclusion

In this paper, the goal of the study is to elaborate the mathematics model for describing the X-ray propagating via imperfect crystals under the non-locality interaction of the X-ray wave field with atoms of crystal medium that probably can be important for digital decoding the nm-scale crystal defects in the computer X-ray diffraction microtomography (cf., [1]). The Cauchy problem of the 2D fractional X-ray diffraction optics designed to describe the mathematical model of the X-ray propagation via the imperfect crystal has been described in terms of the matrix integral Fredholm–Volterra equation. The matrix Resolvent solution of the Cauchy problem in the 2D fractional X-ray diffraction optics has been built and analyzed for the case of the coherent two-beam X-ray diffraction by imperfect crystals under the non-locality interaction of the X-ray with atoms of crystal medium along the crystal thickness. It is shown that in the case, when the fractional order parameter (FOP) $\alpha = 1$, the results obtained have been directly suitable to the mathematical model used of the 2D standard X-ray diffraction optics used in the computer X-ray diffraction microtomography (cf. [7]). Solving the fractional integral-derivative Cauchy problem above presented should be considered as some attempt to take into account the non-locality of the X-photon-atoms interaction in theory of X-ray diffraction crystal microtomography of crystals. To be noticed, the further development and improvement of the theory are a good topic for future work.

Author Contributions

The authors confirm contribution to the paper as follows: study conception and design: Murat O. Mamchuev, Felix N. Chukhovskii; analysis and interpretation of results: Murat O. Mamchuev, Felix N. Chukhovskii; draft manuscript preparation: Murat O. Mamchuev, Felix N. Chukhovskii. All authors reviewed the results and approved the final version of the manuscript.

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Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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