Scientific Eminence Group

Review Article

Open Access

World Journal of Advances in Applied Physics and Mathematical Theories

japm@scientificeminencegroup.com

Vertical Flight with Variable Thrust in Homogeneous Atmosphere

SN Maitra*

Head of Mathematics Department, National Defense Academy, Khadakwasla, Pune -411023(MS), India

***Corresponding Author**

SN Maitra, Head of Mathematics Department, National Defense Academy, Khadakwasla, Pune -411023(MS), India, Email: soumen_maitra@yahoo.co.in

Citation

SN Maitra (2025) Vertical Flight with Variable Thrust in Homogeneous Atmosphere. World J Adv Appl Phys Math Theo 1: 103

Publication Dates

Received date: April 02, 2025
Accepted date: April 18, 2025
Published date: April 25, 2025

Abstract

A rocket/ space vehicle is constrained to fly vertically in atmosphere and gravitational space, in the first case with uniform acceleration with the help of a variable thrust program and in the second case with exponentially time-varying velocity. In either case endurance, ie, time of flight, burnout velocity, burnout altitude ,over-all altitude attained and the controlled thrust are determined. Also evaluated is the propellant mass consumption. Finally some more problems of variable thrust are posed and solved.

Keywords: Homogenous Atmosphere

Background

Angelo Miele [1] analyzed various kinds of flight including vertical flight of rocket with constant thrust and thereafter with constant thrust-to-weight ratio. He [1] dwelt with performance of sounding rocket which soars into space reaching a high altitude overcoming atmospheric drag and gravitational pull where air density decreases as the height increases obeying exponential law and incorporated optimum thrust programming for vertical flight of a rocket.Shaver [2], R.D solved a problem of two-stage sounding rocket.SN Maitra [3] (present author)innovated and solved a problem of vertical flight of a multistage rocket in vacuum maneuvering the same constant thrust in all the stages.SN Maitra[4] worked out analytical solutions to rocket performance in vacuum with an arbitrary thrust program having a constant thrust inclination with respect to the horizon.

In this paper a variable thrust is programmed in such a way that the rocket is launched with an initial velocity from the Earth's surface and travels vertically upwards with a uniform acceleration, obviously overcoming the drag and Page 2 gravitational force. Thus the equations[1] of motion of the rocket for vertical flight are written as

$$m\frac{dV}{dt} = T\text{-}D\text{-}mg \qquad (1)$$

$$T = -V_E\frac{dm}{dt} \qquad (2)$$

$$D = K_D V^2 \qquad (3)$$

$$\frac{dh}{dt} = V$$

$$\frac{dV}{dt} = f \qquad (4)$$

where V denotes the velocity, f the acceleration, m the mass ,T the thrust and h the height at time t, m_0 the initial mass of the rocket, D the drag, V_E the exit velocity

ie,the velocity with which gases are ejected from the nozzles of the rocket, K_D the drag factor and g the acceleration due to gravity and V_0 the initial velocity. Equation (4) subject to the initial conditions as mentioned above yields

$$V = V_0 + ft,\quad h = V_0 t + \frac{1}{2}ft^2 \qquad (5)$$

Equations of Motion With Uniform Acceleration

Combining equations (1) to (4), we get

$$V_E\frac{dm}{dt} = -mf - mg - K_D V^2 \qquad (6)$$

Combining (4) and (6) and simplifying is obtained

$$\frac{dm}{dV} + \left(1 + \frac{g}{f}\right)\frac{m}{V_E} = -\frac{K_D V^2}{fV_E} \qquad (7)$$

which can be solved in conformity with the initial and burnout conditions

At t=0, m= m_0, V=V_0 and h = 0. (8)

At t=t_b,m = m_b, V = V_b, h = h_b

Page3

With $a = \left(1 + \frac{g}{f}\right)\frac{1}{V_E}$ and $A = \frac{K_D}{fV_E}$, (9)

equation (7) reduces to the form

$$\frac{dm}{dV} + am = -AV^2 \qquad (10)$$

Solutions to the Equations Leading to Burnout Conditions

To solve equation (10) we require an integrating factor

$$\text{I.F.} = e^{\int a\,dV} = e^{aV} \qquad (11)$$

Multiplying (10) by (11) and taking up the solution to (10) utilizing the initial conditions (8) is obtained

$$\frac{d(me^{aV})}{dV} = -AV^2e^{aV}$$

$$me^{aV} - m_0e^{aV_0} = -A\int_{V_0}^{V} V^2e^{aV}dv = -A\left\{\frac{V^2e^{aV} - V_0^2e^{aV_0}}{a} - \int_{V_0}^{V} e^{aV}\frac{2V}{a}dV\right\}$$

(Integrating by parts)

$$= -A\left[\frac{V^2e^{aV} - V_0^2e^{aV_0}}{a} - \frac{2}{a^2}\{Ve^{aV} - V_0e^{aV_0} - \int_{V_0}^{V} e^{aV}dV\}\right]$$

$$= -A\left[\frac{V^2e^{aV} - V_0^2e^{aV_0}}{a} - \frac{2}{a^2}\left\{Ve^{aV} - V_0e^{aV_0} - \frac{e^{aV} - e^{aV_0}}{a}\right\}\right]$$

$$m = m_0e^{a(V_0 - V)} - A\left[\frac{V^2e^{aV} - V_0^2e^{aV_0}}{a} - \frac{2}{a^2}\left\{Ve^{aV} - V_0e^{aV_0} - \frac{e^{aV} - e^{aV_0}}{a}\right\}\right]e^{-aV} \qquad (12)$$

Or, using (5) one gets

$$m=m_0e^{-aft}-A\left[\frac{V^2-V_0^2e^{-aft}}{a}-\frac{2}{a^2}\left\{V-V_0e^{-aft}-\frac{1-e^{-ft}}{a}\right\}\right]$$

$$m=m_0e^{-aft}-A\left[\frac{(V_0+ft)^2-V_0^2e^{-aft}}{a}-\frac{2}{a^2}\left\{V_0(1-e^{-aft})+ft-\frac{1-e^{-ft}}{a}\right\}\right] \quad (13)$$

which gives the mass-variation law ie mass of the rocket at time t due to page 4 propellant consumption during the flight with uniform acceleration. Therefore, the propellant mass consumption at time t is given by

$$\rho(t)=m_0-m=$$
$$m_0\{1-e^{-aft}\}+A\left[\frac{(V_0+ft)^2-V_0^2e^{-aft}}{a}-\frac{2}{a^2}\left\{V_0(1-e^{-aft})+ft-\frac{1-e^{-ft}}{a}\right\}\right] \quad (14)$$

Once knowing the propellant mass

$\rho(t_b)$, burnout time t_b and burnout mass m_b can be found out from equation (14).

Combining (2) ,(13) and (6),one gets the instantaneous time-varying thrust

$$T(t)=m(g+f)+K_DV^2$$
$$-A\left[\frac{(V_0+ft)^2-V_0^2e^{-aft}}{a}-\frac{2}{a^2}\left\{V_0(1-e^{-aft})+ft-\frac{1-e^{-ft}}{a}\right\}\right]n_0e^{-aft}$$
$$+K_D(V_0+ft)^2 \quad a>1 \quad (15)$$

$$V_b=V_0+ft_b \quad \text{and} \quad h_b=V_0t_b+\frac{1}{2}ft_b^2 \quad (16)$$

The rocket rises further overcoming atmospheric drag and gravitational pull till it comes to rest. Hence , further height attained by the rocket is due to no thrust but due to the burnout velocity as its initial velocity that helps further upward flight against gravitational and atmospheric resistances for which the relevant equations of motion are

$$\frac{dv}{dt}=\frac{d^2h}{dt^2}=v\frac{dv}{dh}=-\left(\frac{K_Dv^2}{m_b}+g\right) \quad (17)$$

Page 5

where v and h are velocity and altitude after time t of its reaching the burnout position and m_b the mass of the rocket at burnout position followed by closure of the propellant chamber ie with the rocket engine shut off. Hence solution to equation (17) employing thereof initial conditions

At t=0, $v=v_b$, $m=m_b$ and $h=0$: (18)

$$\frac{dv}{dt}=-\frac{K_D}{m_b}\left(v^2+\frac{m_b}{K_D}g\right) \quad (19)$$

$$\text{Or, } \frac{K_D}{m_b}t=\sqrt{\frac{K_D}{gm_b}}\left\{\left(\tan^{-1}\left(v_b\sqrt{\frac{K_D}{gm_b}}\right)-\tan^{-1}\left(v\sqrt{\frac{K_D}{gm_b}}\right)\right\}$$

$$\text{Or, } t=\sqrt{\frac{m_b}{gK_D}}\left\{\left(\tan^{-1}\left(v_b\sqrt{\frac{K_D}{gm_b}}\right)-\tan^{-1}\left(v\sqrt{\frac{K_D}{gm_b}}\right)\right\}\right. \quad (20)$$

$$\text{Or, } \tan^{-1}\left(v\sqrt{\frac{K_D}{gm_b}}\right)=\delta-\sqrt{\frac{gK_D}{m_b}}t \quad \text{where } \delta=\tan^{-1}\left(v_b\sqrt{\frac{K_D}{gm_b}}\right)$$

(20.1)

$$\text{Or, } \frac{dh}{dt}=v=\sqrt{\frac{gm_b}{K_D}}\tan\left(\delta-\sqrt{\frac{gK_D}{m_b}}t\right)$$

$$\text{Or, } v=\sqrt{\frac{gm_b}{K_D}}\frac{\left(\sqrt{\frac{K_D}{gm_b}}\right)v_b-\tan\left(\sqrt{\frac{gK_D}{m_b}}t\right)}{1+\tan\left(\sqrt{\frac{gK_D}{m_b}}t\right)\left(\sqrt{\frac{K_D}{gm_b}}\right)v_b} \quad (21)$$

which reveals the velocity at any instant of time t. Further integration subject to the initial conditions (18) gives the altitude at time t:

$$h=\frac{m_b}{K_D}\left\{\log\cos\left(\delta-\sqrt{\frac{gK_D}{m_b}}t\right)-\log\cos\delta\right\}$$

$$=\frac{m_b}{K_D}\log\frac{\cos\left(\delta-\sqrt{\frac{gK_D}{m_b}}t\right)}{\cos\delta} \quad \text{(Because of(21))}$$

$$\text{Or, } h=\frac{m_b}{K_D}\log\left\{\cos\sqrt{\frac{gK_D}{m_b}}t+\sqrt{\frac{K_D}{gm_b}}v_b\sin\left(\sqrt{\frac{gK_D}{m_b}}t\right)\right\} \quad (22)$$

Page 6

Otherwise h as a function of v is obtained from (17):

$$dh=-\frac{vdv}{\frac{K_Dv^2}{m_b}+g}$$

$$\text{Or, } h=\frac{m_b}{2K_D}\log\frac{v_b^2+\frac{m_b}{K_D}g}{v^2+\frac{m_b}{K_D}g} \quad (23)$$

Now putting v=0 in (20) and(21) and combining with (16) are obtained the total time of flight t_f and the overall height h_f attained as

$$t_f=t_b+\sqrt{\frac{m_b}{gK_D}}\left(\tan^{-1}\left(v_b\sqrt{\frac{K_D}{gm_b}}\right)\right.$$

$$\text{Or, } t_f = t_b + \sqrt{\frac{m_b}{gK_D}}\{\tan^{-1}(V_0 + ft_b)\sqrt{\frac{K_D}{gm_b}}\} \quad (24)$$

$$h_f = = V_0 t_b + \frac{1}{2} f t_b^2 + \frac{m_b}{2K_D}\log(1 + \frac{(V_0 + ft_b)^2 K_D}{m_b g}) \quad (25)$$

Thrust/Mass- Variation Law For Space Vehicle Causing Exponential Increase in Velocity During Vertical Flight in Atmosphere

In this section accomplishing a time-varying thrust program that makes the velocity increase exponentially with respect to time during flight overcoming atmospheric and gravitational force, we have with as usual notations:

$$v = v_0 e^{\alpha t}, \quad \frac{dv}{dt} = v_0 \alpha e^{\alpha t} = \alpha v \quad (26)$$

Where α is constant V_0 the initial velocity and v the velocity at time t .

Hence the equation of motion of the rocket is given by

$$m\frac{dv}{dt} = T - mg - K_D v^2 \quad (27)$$

Equating (27) to (26) and putting the appropriate expression (2) for thrust T and simplifying is obtained

Page7

$$\frac{dm}{dv} + \frac{1}{v_E}\left(1 + \frac{g}{\alpha v}\right) m = \frac{-K_D v}{\alpha v_E} \quad (28)$$

$$\text{I.F.} = e^{\int \frac{1}{v_E}\left(1 + \frac{g}{\alpha v}\right)dv} = e^{\frac{1}{v_E}\left(v + \frac{g}{\alpha}\log v\right)} = e^{\frac{v}{v_E}} v^{g/v_E \alpha}$$

Hence solution to equation (28) with the initial conditions At t=0, v=v_0 , m=m_0 and h=0,

$$m e^{\frac{v}{v_E}}(v)^{\frac{g}{\alpha v_E}} - m_0 e^{\frac{v_0}{v_E}}(v_0)^{\frac{g}{\alpha v_E}} = \frac{-K_D}{\alpha v_E}\int_{v_0}^{v} e^{v/V_E}(v)^{\left(1 + \frac{g}{\alpha V_E}\right)} dv \quad (29)$$

(Integrating by parts)

$$\frac{-K_D}{\alpha}[e^{\frac{v}{V_E}}(v)^{\left(1 + \frac{g}{\alpha V_E}\right)} - e^{\frac{v_0}{V_E}}(v_0)^{\left(1 + \frac{g}{\alpha V_E}\right)} - (1 + \frac{g}{\alpha V_E})\int_{v_0}^{v} e^{v/V_E}(v)^{\frac{g}{\alpha V_E}} / v_E \left(v e^{\frac{v}{V_E}} - v_0 e^{\frac{v_0}{V_E}}\right) - 2\int_{v_0}^{v} V_E e^{\frac{v}{V_E}} dv\}$$

dv]

$$\frac{K_D}{\alpha}[e^{\frac{v}{V_E}}(v_0)^{\left(1 + \frac{g}{\alpha V_E}\right)} - e^{\frac{v_0}{V_E}}(v)^{\left(1 + \frac{g}{\alpha V_E}\right)} + (1 + \frac{g}{\alpha V_E})\int_{v_0}^{v} e^{v/V_E}(v)^{\frac{g}{\alpha V_E}}$$

dv]

The above integral cannot be evaluated in close form but can be evaluated by term-by-term integration after expanding the exponential function e^{v/V_E} in infinite series leading to

$$m e^{\frac{v}{v_E}}(v)^{\frac{g}{\alpha v_E}} - m_0 e^{\frac{v_0}{v_E}}(v_0)^{\frac{g}{\alpha v_E}}$$

$$= \frac{K_D}{\alpha}[e^{\frac{v}{V_E}}(v_0)^{\left(1 + \frac{g}{\alpha V_E}\right)} - e^{\frac{v_0}{V_E}}(v)^{\left(1 + \frac{g}{\alpha V_E}\right)}$$

$$+(1 + \frac{g}{\alpha V_E})\int_{v_0}^{v}\{v^{\frac{g}{\alpha V_E}} + \sum_{r=1}^{\infty}\frac{1}{<r}(\frac{v}{V_E})^r (v)^{\frac{g}{\alpha V_E}}\}dv \quad (30)$$

Or, m

$$e^{\frac{v}{v_E}}(v)^{\frac{g}{\alpha v_E}} = m_0 e^{\frac{v_0}{v_E}}(v_0)^{\frac{g}{\alpha v_E}} + \frac{K_D}{\alpha}[e^{\frac{v}{V_E}}(v_0)^{\left(1 + \frac{g}{\alpha V_E}\right)} - e^{\frac{v_0}{V_E}}(v)^{\left(1 + \frac{g}{\alpha V_E}\right)}$$

$$+\left(1 + \frac{g}{\alpha V_E}\right)\{\frac{(v)^{\left(1 + \frac{g}{\alpha V_E}\right)}}{1 + \frac{g}{\alpha V_E}} - \frac{(v_0)^{\left(1 + \frac{g}{\alpha V_E}\right)}}{1 + \frac{g}{\alpha V_E}} +$$

$$\sum_{r=1}^{r=\infty}(\frac{v^{(r+1+\frac{g}{\alpha V_E})}}{!r(v_E)^r(r+1+\frac{g}{\alpha V_E})} - \frac{v_0^{(r+1+\frac{g}{\alpha V_E})}}{!r(v_E)^r(r+1+\frac{g}{\alpha V_E})})] \quad (31)$$

which gives the mass-variation law for flying with exponentially increasing velocity.

If α in (29) is adjusted as

Page8

$$\alpha = \frac{g}{V_E}, v = v_0 e^{gt/V_E} \quad (32)$$

then it evolves close form solutions:

$$m e^{\frac{v}{v_E}} v - m_0 e^{\frac{v_0}{v_E}} v_0 = \frac{V_E K_D}{g}(e^{\frac{v_0}{v_E}} {v_0}^2 - e^{\frac{v}{v_E}} v^2 + 2\int_{v_0}^{v} e^{\frac{v}{v_E}} v dv)$$

(Integrating parts)

$$m e^{\frac{v}{v_E}} v - m_0 e^{\frac{v_0}{v_E}} v_0 = \frac{V_E K_D}{g}\{e^{\frac{v_0}{v_E}} {v_0}^2 - e^{\frac{v}{v_E}} v^2 + 2$$

$$= \frac{V_E K_D}{g}\{e^{\frac{v_0}{v_E}} {v_0}^2 - e^{\frac{v}{v_E}} v^2 + 2V_E(v e^{\frac{v}{V_E}} - v_0 e^{\frac{v_0}{v_E}}) - 2{V_E}^2(e^{\frac{v}{V_E}} - e^{\frac{v_0}{v_E}}\}$$

or,

$$m(v) = \frac{m_0 v_0}{v} e^{-\frac{v - v_0}{V_E}} + \frac{V_E K_D}{gv}\{(2V_E v - 2{V_E}^2 - v^2)e^{\frac{v}{V_E}}$$

$$-\left(2v_0V_E - 2V_E^2 - v_0^2\right)\frac{1}{v}e^{\frac{v_0}{v_E}}\}e^{-\frac{v}{v_E}} \quad (33)$$

Using (24) and(25), one gets mass of the rocket at time t:

$$m_0e^{-gt/V_E}e^{\frac{-v_0}{V_E}(e^{gt/V_E}-1)} + \frac{V_EK_D}{g}\{\left(2V_E - \frac{2V_E^2e^{-gt/V_E}}{v_0} - v_0e^{gt/V_E}\right)e^{\frac{v_0e^{gt/V_E}}{V_E}}$$

$$-\left(2V_E - 2V_E^2.\frac{1}{v_0} - v_0\right)e^{\frac{v_0-gt}{V_E}}\}e^{-\frac{v_0e^{gt/V_E}}{V_E}} \quad (34)$$

which stands for mass-variation law under the situation mentioned above.

Hence the propellant consumption m_p at time t becomes

$$m_p = m_0 - m$$

$$-e^{\frac{-gt}{V_E}}e^{\frac{-v_0}{V_E}(e^{gt/V_E}-1)}) - \frac{V_EK_D}{g}\{(2V_Ee^{gt/V_E} - 2V_E^2/v_0 - v_0e^{gt/V_E})e^{\frac{v_0e^{gt/V_E}}{V_E}}$$

$$-\left(2v_0V_E - 2V_E^2 - v_0^2\right)e^{\frac{v_0}{V_E}}\}e^{-\frac{v_0e^{gt/V_E}}{V_E}} \quad (35)$$

Page 9

Solving (26) utilizing the initial conditions (28) is obtained the height h(t) attained $h = \frac{v_0}{\alpha}(e^{\alpha t} - 1)$ (36)

In case of situation (30) the height risen is

$$h' = \frac{V_Ev_0}{g}\left(e^{gt/V_E} - 1\right) \quad (37)$$

In consequence of (26),(27)and (34),the thrust can be expressed as function of time t:

$$m_0e^{-gt/V_E}e^{\frac{-v_0}{V_E}(e^{gt/V_E}-1)} + \frac{V_EK_D}{g}\{\left(2V_E - \frac{2V_E^2e^{-gt/V_E}}{v_0} - v_0e^{gt/V_E}\right)e^{\frac{v_0e^{gt/V_E}}{V_E}}$$

$$-\left(2V_E - 2V_E^2.\frac{1}{v_0} - v_0\right)e^{\frac{v_0-gt}{V_E}}\}e^{-\frac{v_0e^{gt/V_E}}{V_E}}]\left(\frac{v_0}{V_E}e^{gt/V_E} + 1\right)g + K_Dv_0^2gt/V_E$$

(38)

Therefore , in order to fly the rocket vertically upwards with exponentially increasing velocity/acceleration with respect to time ,the thrust program has to be maneuvered according to equation(38) and the consequent rate of propellant consumption is

$$\beta(t) = \frac{T(t)}{V_E} \quad (39)$$

Burnout Conditions with Exponentially Time-Varying Velocity

If t_b be time taken by the rocket to exhaust the propellant mass or otherwise the time reckoned to shut off the rocket engine, let us denote the velocity, rocket mass and the propellant mass consumed at that instant respectively by v_b called burnout velocity ,m_b and m_{pb}. Then burnout time t_b can be found out by graphical method or otherwise, given the numerical values, from (30) which purports : at $t=t_b$, $v=v_b$,$m= m_b$ and $m_p= m_{pb}$. In view of the forgoing analysis and in the light of relationship (31) and (22), the rocket,till the propellant is exhausted or the engine is shut off, gains height and velocity so that

Page 10

$$h_b = \frac{v_0}{\alpha}(e^{t_b} - 1) \quad (40)$$

$$v_b = v_0e^{\alpha t_b}$$

Thereafter it will continue to ascend with the help of its initial velocity v_b given by (40) against gravity and atmospheric drag till it comes to rest and the subsequent equations of motion and their solutions can be obtained as are available in the previous section refereeing to (20) and (21).Hence in this case the greatest height attained and the total time taken for the same are given by

$$h' = h_b + \frac{m_b}{2K_D}\log\left(1 + \frac{v_b^2K_D}{m_bg}\right) =$$

$$\frac{v_0}{\alpha}(e^{t_b} - 1) + \frac{m_b}{2K_D}\log\left(1 + \frac{v_0^2e^{2\alpha t_b}K_D}{m_bg}\right) \quad (41)$$

$$t' = t_b + \sqrt{\frac{m_b}{gK_D}}\{\tan^{-1}\left(v_0e^{\alpha t_b}\right)\sqrt{\frac{K_D}{gm_b}}\} \quad (42)$$

where m_b can be obtained in terms of v_b which in turn in terms of t_b using equation(29),vis-à-vis,replacing m and t respectively by m_b and t_b .Or otherwise m_b can be estimated as the gross mass of the rocket minus the propellant mass intake if the same is consumed or minus the propellant mass consumed at the time of the rocket engine having been shut off.

Motion of Rocket with Uniform Velocity During Propellant Consumption

In this mission conforming to as usual notations in the light of preceding text, equations of motion of the rocket with velocity v, thrust T and height h at time t can be set forth as

$$\frac{dh}{dt} = v \quad (43)$$

$$-V_E \frac{dm}{dt} = T = mg + K_D v^2 \qquad (44)$$

Eliminating t between (43) and (44) we get

Page 11

$$\frac{dm}{dh} = -\frac{mg+K_D v^2}{vV_E} \qquad (45)$$

which is integrated subject to the initial conditions

At t=0, h=0 , m=m_0 (46)

$$h=\frac{vV_E}{g}\log\frac{m_0 g+K_D v^2}{mg+K_D v^2} \quad (v=\text{constant})$$

Or, $$\frac{m_0 g+K_D v^2}{mg+K_D v^2} = e^{\frac{gh}{vV_E}}$$

$$mg + K_D v^2 = (m_0 g + K_D v^2)e^{-\frac{gh}{vV_E}}$$

$$m = m_0 e^{-\frac{gh}{vV_E}} - \frac{1}{g}K_D v^2(1 - e^{-\frac{gh}{vV_E}}) \quad \text{(Because h=vt)}$$

Or, $$m=m_0 e^{-\frac{gt}{V_E}} - \frac{1}{g}K_D v^2(1 - e^{-\frac{gt}{V_E}}) \qquad (47)$$

which is the mass -variation law for flying the rocket with uniform velocity v and the propellant mass consumed at that time t is obtained as

$$\rho = m_0 - m = m_0\left(1 - e^{-\frac{gt}{V_E}}\right) + \frac{1}{g}K_D v^2(1 - e^{-\frac{gt}{V_E}}) \qquad (48)$$

Hence the time-varying thrust is obtained in consequence of (44) and (47)as

$$T=(m_0 g + K_D v^2)e^{-\frac{gt}{V_E}} \qquad (48.1)$$

Rocket Flight with Thrust Held Equal to Drag

If thrust is held equal to the drag, with as usual notations

$$T=-V_E \frac{dm}{dt} = K_D V^2 \qquad (49)$$

Or, $V=V_0 - gt$ (50)

Substituting (50) into (49), we get

Page12

$$\frac{dm}{dt} = -\frac{1}{V_E}K_D(V_0 - gt)^2 \qquad (51)$$

Or, $$m = m_0 - \frac{1}{3gV_E}K_D\{V_0^3 - (V_0 - gt)^3\} \qquad (52)$$

Or, $$m = m_0 - \frac{1}{3gV_E}K_D\{3V_0 gt(V_0 - gt) + (gt)^3\} \qquad (53)$$

Hence the propellant mass consumption at any instant of time

t: $$\rho_{T=D}=\frac{1}{3gV_E}K_D\{3V_0 gt(V_0 - gt) + (gt)^3\} \qquad (54)$$

gives the mass–variation law. From (53) t can be expressed as a function of m: $3V_0 gt(V_0 - gt) + (gt)^3 = 3gV_E(m_0 - m)/K_D$

Or, $(V_0 - gt)^3 = V_0^3 - 3gV_E(m_0 - m)/K_D$

Or, $$t = \frac{V_0 - \sqrt[3]{V_0^3 - 3gV_E(m_0 - m)/K_D}}{g} = \frac{V_0 - \sqrt[3]{V_0^3 - 3gV_E(\rho_{T=D})/K_D}}{g}$$

(55)

As earlier, the propellant intake or the propellant mass consumed with the engine shut off is obtained from (54) replacing t by t_b and $\rho_{T=D}$ by ρ_b and mass m of the rocket by m_b,, Owing to equation (49)/(50) the time-varying thrust to be executed is

$$T=K_D(V_0 - gt)^2 \qquad \text{Or, } t = (V_0 - \sqrt{\frac{T}{K_D}})/g \qquad (56)$$

Following the foregoing analysis ,burning time, burnout mass, propellant mass consumed, burnout velocity, burnout altitude are given by

$$t_b = \frac{V_0 - \sqrt[3]{V_0^3 - 3g\rho_b V_E/K_D}}{g} \qquad (57)$$

$$m_b = m_0 - \frac{1}{3gV_E}K_D\{3V_0 gt_b(V_0 - gt_b) + (gt_b)^3\} \qquad (58)$$

$$\rho_b=\frac{1}{3gV_E}K_D\{3V_0 g(V_0 - gt_b) + (gt_b)^3\} \qquad (59)$$

Page13

$$V_b = V_0 - gt_b,\ h_b=V_0 t_b - \frac{1}{2}gt_b^2 \qquad (60)$$

After reaching the burnout position the movement continues in the same direction against gravity and atmospheric drag with the initial velocity and height as above till its velocity vanishes and as such the velocity, height and time of flight after time t of arriving at burnout position are obtained recalling equations(17) to (23.

Motion of Rocket with Thrust Equated to Gravitational Force

Herein the equations of motion applying thrust equal to the gravitational force are presented with as usual notations as

$$T=-V_E\frac{dm}{dt}=mg \qquad (61)$$

$$m\frac{dv}{dt}=T-mg-K_Dv^2=-K_Dv^2 \qquad (62)$$

$$\text{Or, } \frac{dv}{dm}=\frac{K_DV_Ev^2}{m^2g} \qquad (63)$$

Integrating (63) and (61) resorting to the initial conditions: at time t =0, velocity v=v_0, mass $m=m_0$, h=0, the instantaneous velocity, mass, propellant mass consumption ρ and height of the rocket are given by

$$K_DV_E\left(\frac{1}{m}-\frac{1}{m_0}\right)=g\left(\frac{1}{v}-\frac{1}{v_0}\right) \qquad (64)$$

$$m=m_0e^{\frac{-gt}{V_E}},\ \rho=m_0\left(1-e^{\frac{-gt}{V_E}}\right) \qquad (65)$$

Due to (61) and (65),the controlled thrust is exponentially decreasing function of time t:

$$T=m_0ge^{\frac{-gt}{V_E}} \qquad (66)$$

Page 14

Eliminating m between (62) and (65), solving the resulting equation subject to the initial conditions as above, we get

$$\frac{dv}{dt}=\frac{-K_Dv^2}{m_0}e^{\frac{gt}{V_E}}$$

$$\text{Or, } \frac{1}{v}=\frac{1}{v_0}+\frac{K_DV_E}{m_0g}\left(e^{\frac{gt}{V_E}}-1\right) \qquad (67)$$

$$\text{Or, } v=\frac{dh}{dt}=\frac{v_0m_0g}{m_0g+v_0K_DV_E\left(e^{\frac{gt}{V_E}}-1\right)}=\frac{v_0m_0ge^{\frac{-gt}{V_E}}}{(m_0g-v_0K_DV_E)e^{\frac{-gt}{V_E}}+v_0K_DV_E} \qquad (68)$$

$$\text{Or, } h=\frac{V_Ev_0m_0}{(m_0g-v_0V_EK_D)}\log\left(\frac{m_0g}{(m_0g-v_0K_DV_E)e^{\frac{-gt}{V_E}}+v_0K_DV_E}\right)$$

$$=\frac{V_Ev_0m_0}{(m_0g-v_0V_EK_D}\log\left(\frac{1}{\left(1-\frac{v_0K_DV_E}{m_0g}\right)e^{\frac{-gt}{V_E}}+\frac{v_0K_DV_E}{m_0g}}\right)$$

$$\text{Or, } h=\frac{V_Ev_0m_0}{m_0g-v_0V_EK_D}\log\left(\frac{1}{e^{\frac{-gt}{V_E}}+\frac{v_0K_DV_E}{m_0g}\left(1-e^{\frac{-gt}{V_E}}\right)}\right) \qquad (69)$$

$$e^{\frac{-gt}{V_E}}+\frac{v_0K_DV_E}{m_0g}\left(1-e^{\frac{-gt}{V_E}}\right)<1 \quad \text{so that } v_0K_DV_E<m_0g$$

which ensures h>0.

Given the propellant mass consumption ρ_b ,from equation from (65),the burning time is obtained as

$$t_b=\frac{V_E}{g}\log\frac{m_0}{(m_0-\rho_b)} \qquad (70)$$

which can be substituted in (68) and (69); burnout velocity v_b and burnout altitude h_b can be determined. As earlier in line with some relevant foregoing equations, the total time of flight and the greatest height of the rocket can also

page15

be found out. Relationship (70) suggests that given the propellant mass and other parameters, greater the initial gross mass m_0 of the rocket, greater is the burning time t_b of the propellant mass. Similarly equation (67) indicates that the instantaneous velocity v is greater when the initial mass m_0 , which is also called lift- off mass,taken in the rocket is greater.

References

1. Angelo Miele (1962) Flight Mechanics, Theory Of Flight Paths, Vol 1, Addison Wesley Publishing Company, Inc Reading, Massachusetts,PP285-287,336--347.

2. Shaver RD (1967) Two-stage Sounding Rocket Problem, Journal Of Spacecraft And Rockets 4: 1310-1315.

3. Maitra SN (1983) Constant Thrust Of A Multistage Rocket In A Vacuum With A Given Initial Thrust-To-Weight Ratio In The First Stage. J Mathem Phy Sci 17: 4.

4. Maitra SN. Analytical Solutions To Rocket Performance With An Arbitrary Thrust Program Having Constant Thrust Inclination With Respect To Horizon In Vacuum.