# Vertical Flight with Variable Thrust in Homogeneous Atmosphere 

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#### Abstract

A rocket/ space vehicle is constrained to fly vertically in atmosphere and gravitational space, in the first case with uniform acceleration with the help of a variable thrust program and in the second case with exponentially timevarying velocity. In either case endurance, ie, time of flight, burnout velocity, burnout altitude ,over-all altitude attained and the controlled thrust are determined. Also evaluated is the propellant mass consumption. Finally some more problems of variable thrust are posed and solved.


Keywords: Homogenous Atmosphere

## Background

Angelo Miele [1] analyzed various kinds of flight including vertical flight of rocket with constant thrust and thereafter with constant thrust-to-weight ratio. He [1] dwelt with performance of sounding rocket which soars into space reaching a high altitude overcoming atmospheric drag and gravitational pull where air density decreases as the height increases obeying exponential law and incorporated optimum thrust programming for vertical flight of a rocket.Shaver [2], R.D solved a problem of two-stage sounding rocket.SN Maitra [3] (present author)innovated and solved a problem of vertical flight of a multistage rocket in vacuum maneuvering the same constant thrust in all the stages.SN Maitra ${ }^{4}$ worked out analytical solutions to rocket performance in vacuum with an arbitrary thrust program having a constant thrust inclination with respect to the horizon.

In this paper a variable thrust is programmed in such a way that the rocket is launched with an initial velocity from the Earth's surface and travels vertically upwards with a uniform acceleration, obviously overcoming the drag and Page 2 gravitational force. Thus the equations ${ }^{1}$ of motion of the rocket for vertical flight are written as
$\mathrm{m} \frac{\mathrm{dV}}{\mathrm{dt}}=\mathrm{T}-\mathrm{D}-\mathrm{mg}$
$T=-V_{E} \frac{d m}{d t}$
$D=K_{D} V^{2}$
$\frac{\mathrm{dh}}{\mathrm{dt}}=\mathrm{V}$
$\frac{d V}{d t}=f$
where V denotes the velocity, f the acceleration, m the mass, T the thrust and $h$ the height at time $t, m_{0}$ the initial mass of the rocket, D the drag, $\mathrm{V}_{\mathrm{E}}$ the exit velocity
ie,the velocity with which gases are ejected from the nozzles of the rocket, $\mathrm{K}_{\mathrm{D}}$ the drag factor and g the acceleration due to gravity and $V_{0}$ the initial velocity. Equation (4) subject to the initial conditions as mentioned above yields
$\mathrm{V}=\mathrm{V}_{0}+\mathrm{ft}, \mathrm{h}=\mathrm{V}_{0} \mathrm{t}+\frac{1}{2} \mathrm{ft}^{2}$

## Equations of Motion With Uniform Acceleration

Combining equations (1) to (4), we get
$V_{\mathrm{E}} \frac{\mathrm{dm}}{\mathrm{dt}}=-\mathrm{mf}-\mathrm{mg}-\mathrm{K}_{\mathrm{D}} \mathrm{V}^{2}$
Combining (4) and (6) and simplifying is obtained

$$
\begin{equation*}
\frac{\mathrm{dm}}{\mathrm{dV}}+\left(1+\frac{\mathrm{g}}{\mathrm{f}}\right) \frac{\mathrm{m}}{\mathrm{~V}_{\mathrm{E}}}=-\frac{\mathrm{K}_{\mathrm{D}} \mathrm{~V}^{2}}{\mathrm{fV}_{\mathrm{E}}} \tag{7}
\end{equation*}
$$

which can be solved in conformity with the initial and burnout conditions

At $\mathrm{t}=0, \mathrm{~m}=\mathrm{m}_{0}, \mathrm{~V}=\mathrm{V}_{0}$ and $\mathrm{h}=0$.

At $\mathrm{t}=\mathrm{t}_{\mathrm{b}}, \mathrm{m}=\mathrm{m}_{\mathrm{b}}, \mathrm{V}=\mathrm{V}_{\mathrm{b}}, \mathrm{h}=\mathrm{h}_{\mathrm{b}}$

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With $\mathrm{a}=\left(1+\frac{\mathrm{g}}{\mathrm{f}}\right) \frac{1}{\mathrm{~V}_{\mathrm{E}}}$ and $\mathrm{A}=\frac{\mathrm{K}_{\mathrm{D}}}{\mathrm{fv}_{\mathrm{E}}}$,
equation (7) reduces to the form
$\frac{d m}{d V}+a m=-A V^{2}$

## Solutions to the Equations Leading to Burnout Conditions

To solve equation (10) we require an integrating factor
I.F. $=e^{\int a d V}=e^{a V}$

Multiplying (10) by (11) and taking up the solution to (10) utilizing the initial conditions (8) is obtained
$\frac{d\left(\mathrm{me}^{\mathrm{av}}\right)}{d V}=-A V^{2} e^{a V}$
$m e^{a V}-m_{0} e^{a V_{0}}=-A \int_{V_{0}}^{V} V^{2} e^{a V} d v=-A\left\{\frac{V^{2} e^{a V}-V_{0}^{2} e^{a V_{0}}}{a}-\int_{V_{0}}^{V} e^{a V} \frac{2 V}{a} d V\right\}$
(Integrating by parts)

$$
\begin{align*}
& =-A\left[\frac{V^{2} e^{a V}-V_{0}^{2} e^{a V_{0}}}{a}-\frac{2}{a^{2}}\left\{V e^{a V}-V_{0} e^{a V_{0}}-\int_{V_{0}}^{V} e^{a V} d V\right\}\right] \\
& =-A\left[\frac{V^{2} e^{a V}-V_{0}^{2} e^{a V_{0}}}{a}-\frac{2}{a^{2}}\left\{V e^{a V}-V_{0} e^{a V_{0}}-\frac{e^{a V}-e^{a V_{0}}}{a}\right\}\right] \\
& m=m_{0} e^{a\left(V_{0}-V\right)}-A\left[\frac{V^{2} e^{a V}-V V_{0}^{a e^{a}}}{a}-\frac{2}{a^{2}}\left\{V^{a V}-V_{0} e^{a V_{0}}-\frac{e^{a V}-e^{a V}}{a}\right\}\right] e^{-a V} \tag{12}
\end{align*}
$$

Or, using (5) one gets

$$
\begin{align*}
\mathrm{m} & =\mathrm{m}_{0} \mathrm{e}^{-a f t}-\mathrm{A}\left[\frac{V^{2}-V_{0}^{2} e^{-a f t}}{a}-\frac{2}{a^{2}}\left\{V-V_{0} e^{-a f t}-\frac{1-e^{-f t}}{a}\right\}\right] \\
\mathrm{m} & =\mathrm{m}_{0} \mathrm{e}^{-a f t}-\mathrm{A}\left[\frac{\left(V_{0}+f t\right)^{2}-V_{0}^{2} e^{-a f t}}{a}-\frac{2}{a^{2}}\left\{V_{0}\left(1-e^{-a f t}\right)+\mathrm{ft}-\frac{1-e^{-f t}}{a}\right\}\right] \tag{13}
\end{align*}
$$

which gives the mass-variation law ie mass of the rocket at time $t$ due to page 4 propellant consumption during the flight with uniform acceleration. Therefore, the propellant mass consumption at time $t$ is given by

$$
\begin{align*}
& \rho(t)=m_{0}-m= \\
& \quad m_{0}\left\{1-e^{-a f t}\right\}+A\left[\frac{\left(V_{0}+f t\right)^{2}-V_{0}^{2} e^{-a f t}}{a}-\frac{2}{a^{2}}\left\{V_{0}\left(1-e^{-a f t}\right)+f t-\frac{1-e^{-f t}}{a}\right\}\right] \tag{14}
\end{align*}
$$

Once knowing the propellant mass
$\rho\left(t_{b}\right)$, burnout time $t_{b}$ and burnout mass $m_{b}$ can be found out from equation (14).
Combining (2) ,(13) and (6),one gets the instantaneous timevarying thrust
$\mathrm{T}(\mathrm{t})=\mathrm{m}(\mathrm{g}+\mathrm{f})+K_{D} V^{2}$
$-A\left[\frac{\left.V_{0}+f t\right)^{2}-V_{0}^{2} e^{-a f t}}{a}-\frac{2}{a^{2}}\left\{V_{0}\left(1-e^{-a f t}\right)+f t-\frac{1-e^{-f t}}{a}\right\}\right] n_{0} e^{-a f t}$
$+K_{D}\left(\mathrm{~V}_{0}+\mathrm{ft}\right)^{2} \quad \mathrm{a}>1$
$\mathrm{V}_{\mathrm{b}}=\mathrm{V}_{0}+\mathrm{ft}_{\mathrm{b}}$ and $\mathrm{h}_{\mathrm{b}}=\mathrm{V}_{0} \mathrm{t}_{\mathrm{b}}+\frac{1}{2} \mathrm{f} t_{\mathrm{b}}^{2}$

The rocket rises further overcoming atmospheric drag and gravitational pull till it comes to rest. Hence, further height attained by the rocket is due to no thrust but due to the burnout velocity as its initial velocity that helps further upward flight against gravitational and atmospheric resistances for which the relevant equations of motion are
$\frac{\mathrm{dv}}{\mathrm{dt}}=\frac{d^{2} h}{d t^{2}}=\mathrm{v} \frac{\mathrm{dv}}{\mathrm{dh}}=-\left(\frac{\mathrm{K}_{\mathrm{D}} \mathrm{v}^{2}}{\mathrm{~m}_{\mathrm{b}}}+\mathrm{g}\right)$

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where v and h are velocity and altitude after time t of its reaching the burnout position and $m_{b}$ the mass of the rocket at burnout position followed by closure of the propellant chamber ie with the rocket engine shut off. Hence solution to equation (17) employing thereof initial conditions
At $\mathrm{t}=0, \mathrm{v}=\mathrm{v}_{\mathrm{b}}, \mathrm{m}=\mathrm{m}_{\mathrm{b}}$ and $\mathrm{h}=0$ :
$\frac{\mathrm{dv}}{\mathrm{dt}}=-\frac{\mathrm{K}_{\mathrm{D}}}{\mathrm{m}_{\mathrm{b}}}\left(\mathrm{v}^{2}+\frac{\mathrm{m}_{\mathrm{b}}}{\mathrm{K}_{\mathrm{D}}} \mathrm{g}\right)$
Or, $\left.\left.\frac{\mathrm{K}_{\mathrm{D}}}{\mathrm{m}_{\mathrm{b}}} \mathrm{t}=\sqrt{\frac{\mathrm{K}_{\mathrm{D}}}{g \mathrm{~m}_{\mathrm{b}}}}\left\{\tan ^{-1\left(\mathrm{v}_{\mathrm{b}}\right.} \sqrt{\frac{\mathrm{K}_{\mathrm{D}}}{g \mathrm{~m}_{\mathrm{b}}}}\right)-\tan ^{-(\mathrm{v}} \sqrt{\frac{\mathrm{K}_{\mathrm{D}}}{\mathrm{gm}_{\mathrm{b}}}}\right)\right\}$
Or, $\left.\mathrm{t}=\sqrt{\frac{\mathrm{m}_{\mathrm{b}}}{\mathrm{gK}_{\mathrm{D}}}}\left\{\left(\tan ^{-1\left(\mathrm{v}_{\mathrm{b}}\right.} \sqrt{\frac{\mathrm{K}_{\mathrm{D}}}{\mathrm{gm}_{\mathrm{b}}}}\right)-\tan ^{-1(\mathrm{v}} \sqrt{\frac{\mathrm{K}_{\mathrm{D}}}{\mathrm{gm}_{\mathrm{b}}}}\right)\right\}$

Or, $\left.\tan ^{-(v} \sqrt{\frac{K_{D}}{g m_{\mathrm{b}}}}\right)=\delta-\sqrt{\frac{\mathrm{gK}_{\mathrm{D}}}{\mathrm{m}_{\mathrm{b}}}} \mathrm{t} \quad$ where $\delta=\tan ^{-1\left(\sqrt{\frac{K_{D}}{g m_{\mathrm{b}}}}\right) \mathrm{v}_{\mathrm{b}}}$
(20.1)

Or, $\frac{\mathrm{dh}}{\mathrm{dt}}=\mathrm{v}=\sqrt{\frac{\mathrm{gm}_{\mathrm{b}}}{\mathrm{K}_{\mathrm{D}}}} \tan \left(\delta-\sqrt{\frac{\mathrm{gK}_{\mathrm{D}}}{\mathrm{m}_{\mathrm{b}}}} \mathrm{t}\right)$

Or, $v=\sqrt{\frac{g m_{b}}{K_{D}}} \frac{\left(\sqrt{\frac{K_{D}}{g m_{b}}}\right) v_{b}-\tan \left(\sqrt{\frac{g K_{D}}{m_{b}}} t\right)}{1+\tan \left(\sqrt{\frac{g K_{D}}{m_{b}}} t\right)\left(\sqrt{\frac{K_{D}}{g m_{b}}}\right) v_{b}}$
which reveals the velocity at any instant of time $t$. Further integration subject to the initial conditions (18) gives the altitude at time t :
$\mathrm{h}=\frac{\mathrm{m}_{\mathrm{b}}}{\mathrm{K}_{\mathrm{D}}}\left\{\log \cos \left(\delta-\sqrt{\frac{\mathrm{KK}_{\mathrm{D}}}{\mathrm{m}_{\mathrm{b}}}} \mathrm{t}\right)-\log \cos \delta\right\}$
$=\frac{\mathrm{m}_{\mathrm{b}}}{\mathrm{K}_{\mathrm{D}}} \log \frac{\cos \left(8-\sqrt{\frac{\mathrm{EK}_{\mathrm{D}}}{\mathrm{m}_{\mathrm{b}}}} \mathrm{t}\right)}{\cos \delta}$
(Because of(21))

Or, $\mathrm{h}=\frac{\mathrm{m}_{\mathrm{b}}}{\mathrm{K}_{\mathrm{D}}} \log \left\{\cos \sqrt{\frac{\mathrm{gK}_{\mathrm{D}}}{\mathrm{m}_{\mathrm{b}}}} \mathrm{t}+\sqrt{\frac{\mathrm{K}_{\mathrm{D}}}{\mathrm{gm}_{\mathrm{b}}}} \mathrm{v}_{\mathrm{b}} \sin \left(\sqrt{\frac{\mathrm{gK}_{\mathrm{D}}}{\mathrm{m}_{\mathrm{b}}}} \mathrm{t}\right)\right\}$

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Otherwise $h$ as a function of $v$ is obtained from (17):
$\mathrm{d} h=-\frac{\mathrm{vdv}}{\frac{K_{\mathrm{D}} v^{2}}{\mathrm{~m}_{\mathrm{b}}}+\mathrm{g}}$

Or, $\mathrm{h}=\frac{\mathrm{m}_{\mathrm{b}}}{2 \mathrm{~K}_{\mathrm{D}}} \log \frac{\mathrm{v}_{\mathrm{b}}^{2}+\frac{\mathrm{m}_{\mathrm{b}}}{\mathrm{K}_{\mathrm{D}}}}{\mathrm{v}^{2}+\frac{\mathrm{m}_{\mathrm{b}}}{\mathrm{K}_{\mathrm{D}}} \mathrm{g}}$

Now putting $\mathrm{v}=0$ in (20) and(21) and combining with (16) are obtained the total time of flight $t_{f}$ and the overall height $h_{f}$ attained as
$t_{f}=t_{b}+\sqrt{\frac{m_{b}}{g K_{D}}}\left(\tan ^{-1\left(v_{b}\right.} \sqrt{\frac{K_{D}}{g m_{b}}}\right)$

Or, $\mathrm{t}_{\mathrm{f}}=\mathrm{t}_{\mathrm{b}}+\sqrt{\frac{\mathrm{m}_{\mathrm{b}}}{\mathrm{gK}_{\mathrm{D}}}}\left\{\tan ^{-1\left(\mathrm{~V}_{0}+\mathrm{ft}_{\mathrm{b}}\right)} \sqrt{\frac{\mathrm{K}_{\mathrm{D}}}{\mathrm{gm}_{\mathrm{b}}}}\right\}$
$\mathrm{h}_{\mathrm{f}}==\mathrm{V}_{0} \mathrm{t}_{\mathrm{b}}+\frac{1}{2} \mathrm{ft} t_{\mathrm{b}}^{2}+\frac{\mathrm{m}_{\mathrm{b}}}{2 \mathrm{~K}_{\mathrm{D}}} \log \left(1+\frac{\left(\mathrm{V}_{0}+\mathrm{ft}_{\mathrm{b}}\right)^{2} \mathrm{~K}_{\mathrm{D}}}{\mathrm{m}_{\mathrm{b}} \mathrm{g}}\right)$

Thrust/Mass- Variation Law For Space Vehicle Causing Exponential Increase in Velocity During Vertical Flight in Atmosphere

In this section accomplishing a time-varying thrust program that makes the velocity increase exponentially with respect to time during flight overcoming atmospheric and gravitational force, we have with as usual notations:
$v=v_{0} e^{a t}, \quad \frac{d v}{d t}=v_{0} \alpha e^{a t}=\alpha v$

Where $\alpha$ is constant $V_{0}$ the initial velocity and $v$ the velocity at time $t$.

Hence the equation of motion of the rocket is given by
$\mathrm{m} \frac{\mathrm{dv}}{\mathrm{dt}}=\mathrm{T}-\mathrm{mg}-\mathrm{K}_{\mathrm{D}} \mathrm{v}^{2}$

Equating (27) to (26) and putting the appropriate expression (2) for thrust T and simplifying is obtained

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$\frac{d m}{d v}+\frac{1}{v_{\mathrm{E}}}\left(1+\frac{g}{\alpha v}\right) m=\frac{-\mathrm{K}_{\mathrm{D}} \mathrm{v}}{\alpha v_{\mathrm{E}}}$
I.F. $=e^{\int \frac{1}{v_{\mathrm{E}}}\left(1+\frac{\mathrm{g}}{\alpha \mathrm{v}}\right) d v}=e^{\frac{1}{v_{\mathrm{E}}}\left(v+\frac{g_{\mathrm{a}}}{\alpha} \log v\right)}=e^{\frac{v}{v_{\mathrm{E}}}} v^{\mathrm{g}} / \mathrm{v}_{\mathrm{E}} \alpha$

Hence solution to equation (28) with the initial conditions At $t=0, v=v_{0}, m=m_{0}$ and $h=0$,
$\mathrm{m}_{e^{\frac{v}{\mathrm{~V}_{\mathrm{E}}}}}^{\frac{v}{}}(\mathrm{v})^{\frac{g}{\alpha}} \mathrm{v}_{\mathrm{E}}-\mathrm{m}_{0} e^{\frac{(29)}{\mathrm{v}_{\mathrm{E}}}}\left(\mathrm{v}_{0}\right)^{\frac{g}{a}} \mathrm{~V}_{\mathrm{E}}=\frac{-\mathrm{K}_{\mathrm{D}}}{\alpha \mathrm{v}_{\mathrm{E}}} \int_{\mathrm{V}_{0}}^{v} e^{v / V_{E}}(v)^{\left(1+\frac{\mathrm{g}}{\alpha \mathrm{V}_{\mathrm{E}}}\right)} \mathrm{dv}$
(Integrating by parts)

$$
\mathrm{m} e^{\frac{v}{\mathrm{v}_{\mathrm{E}}}} v-\mathrm{m}_{0} e^{\frac{\mathrm{v}_{0}}{\mathrm{v}_{\mathrm{E}}}} \mathrm{~V}_{0}=\frac{\mathrm{V}_{\mathrm{E}} \mathrm{~K}_{\mathrm{D}}}{\mathrm{~g}}\left\{e^{\frac{\mathrm{v}_{0}}{\mathrm{~V}_{\mathrm{E}}} \mathrm{~V}_{0}}{ }^{2}-e^{\frac{v}{\mathrm{v}_{\mathrm{E}}}} v^{2}+2\right.
$$


dv]
$\frac{\mathrm{K}_{\mathrm{D}}}{\alpha}\left[e^{\frac{v}{V_{E}}}\left(\mathrm{~V}_{0}\right)^{\left(1+\frac{\mathrm{g}}{\alpha V_{\mathrm{E}}}\right.}\right)-e^{\left.\frac{\mathrm{v}_{0}}{V_{E}}(\mathrm{~V})^{\left(1+\frac{\mathrm{g}}{\alpha V_{\mathrm{E}}}\right.}\right)}+\left(1+\frac{\mathrm{g}}{\alpha V_{\mathrm{E}}}\right) \int_{\mathrm{V}_{0}}^{v} e^{\mathrm{v} / \mathrm{V}_{\mathrm{E}}}(v)^{\frac{\mathrm{g}}{\alpha \mathrm{V}_{E}}}$
 or,

$$
m(v)=\frac{m_{0} v_{0}}{v} e^{-\frac{v-v_{0}}{v_{E}}}+\frac{v_{E} K_{D}}{g v}\left\{\left(2 V_{E} v-2 V_{E}^{2}-v^{2}\right) e^{\frac{v}{V_{E}}}\right.
$$

$\left.-\left(2 v_{0} V_{E}-2 V_{E}^{2}-v_{0}^{2}\right) \frac{1}{v} e^{\frac{v_{0}}{v_{E}}}\right\} e^{-\frac{v}{v_{E}}}$

Using (24) and(25), one gets mass of the rocket at time t:

$\left.-\left(2 V_{E}-2 V_{E}^{2} \cdot \frac{1}{v_{0}}-v_{0}\right) e^{\frac{v_{0}-g t}{v_{E}}}\right\} e^{-\frac{v_{0} g g t / V_{E}}{v_{E}}}$
which stands for mass-variation law under the situation mentioned above.

Hence the propellant consumption $\mathrm{m}_{\mathrm{p}}$ at time t becomes
$m_{p}=m_{0}-m$

$\left.-\left(2 v_{0} V_{E}-2 V_{E}^{2}-v_{0}^{2}\right) e^{\frac{v_{0}}{v_{E}}}\right\} e^{-\frac{v_{0} g}{} g^{g} / V_{E}}$
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Solving (26) utilizing the initial conditions (28) is obtained the height $h(t)$ attained $h=\frac{v_{0}}{\alpha}\left(e^{a t}-1\right)$

In case of situation (30) the height risen is
$\mathrm{h}^{\prime}=\frac{V_{E} V_{0}}{g}\left(e^{g t / V_{E}}-1\right)$

In consequence of (26),(27)and (34),the thrust can be expressed as function of time $t$ :
$m_{0} e^{-g t / V_{E}} e^{\frac{-v_{0}}{V_{E}}\left(e^{g t / V_{E}}-1\right)}+\frac{V_{E} K_{D}}{g}\left\{\left(2 V_{E}-\frac{2 V_{E}^{2} e^{-g t / V_{E}}}{v_{0}}-v_{0} e^{g t / V_{E}}\right) e^{\frac{v_{0} e^{g t / V_{E}}}{V_{E}}}\right.$ $\left.-\left(2 V_{\mathrm{E}}-2 V_{\mathrm{E}}^{2} \cdot \frac{1}{v_{0}}-v_{0}\right) e^{\frac{v_{0}-\mathrm{v}_{\mathrm{E}}}{v_{\mathrm{E}}}} 3 e^{-\frac{v_{0} e^{\mathrm{g} t / V_{\mathrm{E}}}}{v_{\mathrm{E}}}}\right]\left(\frac{v_{0}}{V_{\mathrm{E}}} e^{g t / V_{\mathrm{E}}}+1\right) g+\mathrm{K}_{\mathrm{D}} V_{0}^{2 g t / V_{\mathrm{E}}}$

Therefore, in order to fly the rocket vertically upwards with exponentially increasing velocity/acceleration with respect to time ,the thrust program has to be maneuvered according to equation(38) and the consequent rate of propellant consumption
is $\beta(t)=\frac{\tau(t)}{V_{\mathrm{E}}}$

## Burnout Conditions with Exponentially Time-Varying Velocity

If $\mathrm{t}_{\mathrm{b}}$ be time taken by the rocket to exhaust the propellant mass or otherwise the time reckoned to shut off the rocket engine, let us denote the velocity, rocket mass and the propellant mass consumed at that instant respectively by $\mathrm{v}_{\mathrm{b}}$ called burnout velocity, $\mathrm{m}_{\mathrm{b}}$ and $\mathrm{m}_{\mathrm{pb}}$. Then burnout time $\mathrm{t}_{\mathrm{b}} \mathrm{can}$ be found out by graphical method or otherwise, given the numerical values, from (30) which purports : at $\mathrm{t}=\mathrm{t}_{\mathrm{b}}, \mathrm{v}=\mathrm{v}_{\mathrm{b},}, \mathrm{m}=\mathrm{m}_{\mathrm{b}}$ and $\mathrm{m}_{\mathrm{p}}=\mathrm{m}_{\mathrm{pb}}$. In view of the forgoing analysis and in the light of relationship (31) and (22), the rocket,till the propellant is exhausted or the engine is shut off, gains height and velocity so that
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$h_{b}=\frac{v_{0}}{\alpha}\left(\mathrm{e}^{\mathrm{t}_{\mathrm{b}}}-1\right)$
$\mathrm{v}_{\mathrm{b}}=\mathrm{v}_{0} e^{a t_{\mathrm{b}}}$

Thereafter it will continue to ascend with the help of its initial velocity $\mathrm{v}_{\mathrm{b}}$ given by (40) against gravity and atmospheric drag till it comes to rest and the subsequent equations of motion and their solutions can be obtained as are available in the previous section refereeing to (20) and (21).Hence in this case the greatest height attained and the total time taken for the same are given by

$$
\begin{align*}
& \mathrm{h}^{\prime}=\mathrm{h}_{\mathrm{b}}+\frac{\mathrm{m}_{\mathrm{b}}}{2 \mathrm{~K}_{\mathrm{D}}} \log \left(1+\frac{v_{\mathrm{b}}^{2} \mathrm{~K}_{\mathrm{D}}}{\mathrm{~m}_{\mathrm{b}} \mathrm{~g}}\right)= \\
& \frac{v_{0}}{\alpha}\left(\mathrm{e}^{\mathrm{t}_{\mathrm{b}}}-1\right)+\frac{\mathrm{m}_{\mathrm{b}}}{2 \mathrm{~K}_{\mathrm{D}}} \log \left(1+\frac{v_{0}^{2} \mathrm{e}^{2 \alpha t_{\mathrm{b}} K_{\mathrm{D}}}}{\mathrm{~m}_{\mathrm{b}} \mathrm{~g}}\right) \tag{41}
\end{align*}
$$

$\mathrm{t}^{\prime}=\mathrm{t}_{\mathrm{b}}+\sqrt{\frac{\mathrm{m}_{\mathrm{b}}}{\mathrm{gK}_{\mathrm{D}}}}\left\{\tan ^{-1\left(v_{0} e^{\left.\alpha t_{\mathrm{b}}\right)}\right.} \sqrt{\frac{\mathrm{K}_{\mathrm{D}}}{\mathrm{gm}_{\mathrm{b}}}}\right\}$
where $\mathrm{m}_{\mathrm{b}}$ can be obtained in terms of $\mathrm{v}_{\mathrm{b}}$ which in turn in terms of $\mathrm{t}_{\mathrm{b}}$ using equation(29), vis-à-vis,replacing m and t respectively by $\mathrm{m}_{\mathrm{b}}$ and $\mathrm{t}_{\mathrm{b}}$. Or otherwise $\mathrm{m}_{\mathrm{b}}$ can be estimated as the gross mass of the rocket minus the propellant mass intake if the same is consumed or minus the propellant mass consumed at the time of the rocket engine having been shut off.

## Motion of Rocket with Uniform Velocity During Propellant Consumption

In this mission conforming to as usual notations in the light of preceding text, equations of motion of the rocket with velocity v , thrust T and height h at time t can be set forth as
$\frac{\mathrm{dh}}{\mathrm{dt}}=\mathrm{v}$
$-V_{E} \frac{d m}{d t}=T=m g+K_{D} v^{2}$

Eliminating t between (43) and (44) we get

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$\frac{\mathrm{dm}}{\mathrm{dh}}=-\frac{\mathrm{mg}+\mathrm{K}_{\mathrm{D}} \mathrm{v}^{2}}{\mathrm{vV}_{\mathrm{E}}}$
which is integrated subject to the initial conditions

At $\mathrm{t}=0, \mathrm{~h}=0, \mathrm{~m}=\mathrm{m}_{0}$
$h=\frac{v V_{E}}{g} \log \frac{m_{0} g+K_{D} v^{2}}{m g+K_{D} v^{2}} \quad$ ( $\mathrm{v}=$ constant $)$
Or, $\frac{m_{0} g+K_{D} v^{2}}{m g+K_{D} v^{2}}=e^{\frac{g h}{v V_{E}}}$
$\mathrm{mg}+\mathrm{K}_{\mathrm{D}} \mathrm{v}^{2}=\left(\mathrm{m}_{0} \mathrm{~g}+\mathrm{K}_{\mathrm{D}} \mathrm{v}^{2}\right) \mathrm{e}^{-\frac{\mathrm{gh}}{\mathrm{vV}_{E}}}$
$m=m_{0} e^{-\frac{g h}{v V_{E}}}-\frac{1}{g} K_{D} v^{2}\left(1-e^{-\frac{g h}{v V_{E}}}\right) \quad$ (Because $h=v t$ )
Or, $m=m_{0} e^{-\frac{g t}{v_{E}}}-\frac{1}{g} K_{D} v^{2}\left(1-e^{-\frac{g t}{v_{E}}}\right)$
which is the mass -variation law for flying the rocket with uniform velocity v and the propellant mass consumed at that time $t$ is obtained as
$\rho=m_{0}-m=m_{0}\left(1-e^{-\frac{g t}{v_{E}}}\right)+\frac{1}{g} K_{D} v^{2}\left(1-e^{-\frac{g t}{v_{E}}}\right)$

Hence the time-varying thrust is obtained in consequence of (44) and (47)as
$T=\left(m_{0} g+K_{D} v^{2}\right) e^{-\frac{g t}{v_{E}}}$

## Rocket Flight with Thrust Held Equal to Drag

If thrust is held equal to the drag, with as usual notations
$\mathrm{T}=-\mathrm{V}_{\mathrm{E}} \frac{\mathrm{dm}}{\mathrm{dt}}=\mathrm{K}_{\mathrm{D}} \mathrm{V}^{2}$

Or, $V=V_{0}-g t$

Substituting (50) into (49), we get

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$\frac{\mathrm{dm}}{\mathrm{dt}}=-\frac{1}{\mathrm{~V}_{\mathrm{E}}} \mathrm{K}_{\mathrm{D}}\left(\mathrm{V}_{0}-\mathrm{gt}\right)^{2}$
Or, $\mathrm{m}=\mathrm{m}_{0}-\frac{1}{3 g \mathrm{~V}_{\mathrm{E}}} \mathrm{K}_{\mathrm{D}}\left\{\mathrm{V}_{0}{ }^{3}-\left(\mathrm{V}_{0}-\mathrm{gt}\right)^{3}\right\}$
Or, $m=m_{0}-\frac{1}{3 g V_{E}} K_{D}\left\{3 V_{0} g t\left(V_{0}-g t\right)+(g t)^{3}\right\}$

Hence the propellant mass consumption at any instant of time
$\mathrm{t}: \rho_{T=D}=\frac{1}{3 \mathrm{~g} \mathrm{~V}_{\mathrm{E}}} \mathrm{K}_{\mathrm{D}}\left\{3 \mathrm{~V}_{0} \mathrm{gt}\left(\mathrm{V}_{0}-\mathrm{gt}\right)+(\mathrm{gt})^{3}\right\}$
gives the mass-variation law. From (53) t can be expressed as a function of $m: 3 V_{0} g t\left(V_{0}-g t\right)+(g t)^{3}=3 g V_{E}\left(m_{0}-m\right) / K_{D}$

Or, $\left(\mathrm{V}_{0}-\mathrm{gt}\right)^{3}=\mathrm{V}_{0}{ }^{3}-3 \mathrm{gV}_{\mathrm{E}}\left(\mathrm{m}_{0}-m\right) / \mathrm{K}_{\mathrm{D}}$
Or, $\mathrm{t}=\frac{\mathrm{V}_{0}-\sqrt[3]{\mathrm{V}_{0}{ }^{3}-3 \mathrm{~g} \mathrm{~V}_{\mathrm{E}}\left(\mathrm{m}_{0}-m\right) / \mathrm{K}_{\mathrm{D}}}}{g}=\frac{\mathrm{V}_{0}-\sqrt[3]{\mathrm{V}_{0}{ }^{3}-3 \mathrm{~g} \mathrm{~V}_{\mathrm{E}}\left(\rho_{T=D}\right) / \mathrm{K}_{\mathrm{D}}}}{g}$

As earlier, the propellant intake or the propellant mass consumed with the engine shut off is obtained from (54) replacing $t$ by $t_{b}$ and $\rho_{\tau=D}$ by $\rho_{\mathrm{b}}$ and mass m of the rocket by $\mathrm{m}_{\mathrm{b}, \text {, }}$ Owing to equation $(49) /(50)$ the time-varying thrust to be executed is
$\mathrm{T}=\mathrm{K}_{\mathrm{D}}\left(\mathrm{V}_{0}-\mathrm{gt}\right)^{2} \quad$ Or, $\mathrm{t}=\left(\mathrm{V}_{0}-\sqrt{\frac{\tau}{\mathrm{x}_{D}}}\right) / \mathrm{g}$

Following the foregoing analysis ,burning time, burnout mass, propellant mass consumed, burnout velocity, burnout altitude are given by
$\mathrm{t}_{\mathrm{b}}=\frac{\mathrm{V}_{0}-\sqrt[3]{\mathrm{v}_{0}{ }^{3}-3 \mathrm{~g} \mathrm{pb}_{\mathrm{b}} \mathrm{V}_{\mathrm{E}} / \mathrm{K}_{\mathrm{D}}}}{g}$
$\mathrm{m}_{\mathrm{b}}=\mathrm{m}_{0}-\frac{1}{3 \mathrm{~g} \mathrm{~V}_{\mathrm{E}}} \mathrm{K}_{\mathrm{D}}\left\{3 \mathrm{~V}_{0} \mathrm{~g} \mathrm{t}_{\mathrm{b}}\left(\mathrm{V}_{0}-\mathrm{gt}_{\mathrm{b}}\right)+\left(\mathrm{gt}_{\mathrm{b}}\right)^{3}\right\}$
$\rho_{b}=\frac{1}{3 g V_{E}} K_{D}\left\{3 V_{0} g\left(V_{0}-g t_{b}\right)+\left(g t_{b}\right)^{3}\right\}$
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$\mathrm{V}_{\mathrm{b}}=\mathrm{V}_{0}-g \mathrm{t}_{\mathrm{b}}, \mathrm{h}_{\mathrm{b}}=\mathrm{V}_{0} \mathrm{t}_{\mathrm{b}}-\frac{1}{2} g \mathrm{t}_{\mathrm{b}}^{2}$

After reaching the burnout position the movement continues in the same direction against gravity and atmospheric drag with the initial velocity and height as above till its velocity vanishes and as such the velocity, height and time of flight after time $t$ of arriving at burnout position are obtained recalling equations(17) to (23.

## Motion of Rocket with Thrust Equated to Gravitational Force

Herein the equations of motion applying thrust equal to the gravitational force are presented with as usual notations as
$\mathrm{T}=-\mathrm{V}_{\mathrm{E}} \frac{\mathrm{dm}}{\mathrm{dt}}=\mathrm{mg}$
$\mathrm{m} \frac{\mathrm{dv}}{\mathrm{d}^{+}}=\mathrm{T}-\mathrm{mg}-\mathrm{K}_{\mathrm{D}} \mathrm{v}^{2}=-\mathrm{K}_{\mathrm{D}} \mathrm{v}^{2}$
Or, $\frac{d v}{d m}=\frac{K_{D} V_{E} v^{2}}{m^{2} g}$

Integrating (63) and (61) resorting to the initial conditions: at time $t=0$, velocity $\mathrm{v}=\mathrm{v}_{0}$, mass $\mathrm{m}=\mathrm{m}_{0}, \mathrm{~h}=0$, the instantaneous velocity, mass, propellant mass consumption $\rho$ and height of the rocket are given by
$\mathrm{K}_{\mathrm{D}} \mathrm{V}_{\mathrm{E}}\left(\frac{1}{m}-\frac{1}{\mathrm{~m}_{0}}\right)=g\left(\frac{1}{\mathrm{v}}-\frac{1}{\mathrm{v}_{0}}\right)$
$m=m_{0} e^{\frac{-\mathrm{gt}}{\mathrm{V}_{\mathrm{E}}}}, \rho=\mathrm{m}_{0}\left(1-e^{\frac{-\mathrm{gt}}{\mathrm{V}_{\mathrm{E}}}}\right)$

Due to (61) and (65),the controlled thrust is exponentially decreasing function of time $t$ :
$\mathrm{T}=\mathrm{m}_{0} \mathrm{ge}^{\frac{-\mathrm{gt}}{\mathrm{V}_{\mathrm{E}}}}$

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Eliminating m between (62) and (65), solving the resulting equation subject to the initial conditions as above, we get

$$
\begin{align*}
& \frac{d v}{d t}=\frac{-K_{D} v^{2}}{m_{0}} e^{\frac{g t}{V_{\mathrm{E}}}} \\
& \text { Or, } \frac{1}{\mathrm{~V}}=\frac{1}{v_{0}}+\frac{\mathrm{K}_{\mathrm{D}} V_{\mathrm{E}}}{m_{0} g}\left(\mathrm{e}^{\frac{\mathrm{gt}}{\mathrm{~V}_{\mathrm{E}}}}-1\right)  \tag{67}\\
& \text { Or, } \mathrm{v}=\frac{d h}{d t}=\frac{v_{0}}{m_{0} g+\mathrm{v}_{0} K_{D} V_{\mathrm{E}}\left(\mathrm{e}^{\mathrm{gt}} \mathrm{~V}_{\mathrm{E}}-1\right)} \\
& \text { (68) }
\end{align*}
$$

$$
\text { Or, } \mathrm{h}=\frac{\mathrm{V}_{\mathrm{E}} \mathrm{v}_{0} \mathrm{~m}_{0}}{\left(\mathrm{~m}_{0} \mathrm{~g}-\mathrm{v}_{0} \mathrm{~V}_{\mathrm{E}} \mathrm{~K}_{\mathrm{D}}\right)} \log \frac{\mathrm{m}_{0} \mathrm{~g}}{\left(\left(\mathrm{~m}_{0} \mathrm{~g}-\mathrm{v}_{0} \mathrm{~K}_{\mathrm{D}} \mathrm{~V}_{\mathrm{E}}\right) \mathrm{e}^{\frac{-\mathrm{gt}}{\mathrm{~V}_{\mathrm{E}}}+\mathrm{v}_{0} K_{\mathrm{D}} \mathrm{~V}_{\mathrm{E}}}\right)}
$$

$$
=\frac{v_{E} v_{0} m_{0}}{\left(m_{0} g-v_{0} v_{E} K_{D}\right.} \log \left(\left(1-\frac{v_{0} K_{D} v_{E}}{m_{0} g}\right)_{e} \frac{-g t}{v_{E}}+\frac{v_{0} K_{D} V_{E}}{m_{0} g}\right)
$$

Or, $h=\frac{V_{E} v_{0} m_{0}}{m_{0} g-v_{0} V_{E} K_{D}} \log \left(e^{\frac{-g t}{V_{E}}}+\frac{1}{v_{0} K_{D} V_{E}} m_{0}\left(1-e^{\frac{-g t}{V_{E}}}\right), ~\right.$
$e^{\frac{-g t}{V_{E}}}+\frac{v_{0} K_{D} V_{E}}{m_{0} g}\left(1-e^{\frac{-g t}{V_{E}}}\right)<1$ so that $V_{0} K_{D} V_{E}<m_{0} g$
which ensures $\mathrm{h}>0$.

Given the propellant mass consumption $\rho_{\mathrm{b}}$, from equation from (65),the burning time is obtained as
$t_{b}=\frac{V_{\mathrm{E}}}{\mathrm{g}} \log \frac{\mathrm{m}_{0}}{\left(\mathrm{~m}_{0}-\rho_{\mathrm{b}}\right)}$
which can be substituted in (68) and (69); burnout velocity $\mathrm{v}_{\mathrm{b}}$ and burnout altitude $h_{b}$ can be determined. As earlier in line with some relevant foregoing equations, the total time of flight and the greatest height of the rocket can also
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be found out. Relationship (70) suggests that given the propellant mass and other parameters, greater the initial gross mass $m_{0}$ of the rocket, greater is the burning time $\mathrm{t}_{\mathrm{b}}$ of the propellant mass. Similarly equation (67) indicates that the instantaneous velocity $v$ is greater when the initial mass $m_{0}$, which is also called lift- off mass,taken in the rocket is greater.

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